Cooley-Tukey Type Discrete Fourier Transform Algorithm For Continuous Function Sampled At Some Composite Point $N = pq, p \neq q$

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ABSTRACT
By introducing complete base number indices and restricting the number of sampled points to the value N which is highly composite, the paper is focused on the computational complexity of Discrete Fourier Transform (DFT) of a continuous function. We construct a Cooley-Tukey type Fast Fourier Transform FFT algorithm aimed at reducing the number of complex computational operations from N^2 complex multiplications to $Ny/_2$ and from $N(N-1)$ complex additions to Ny, where y is an integer to which the selected base number is raised. To justify the effectiveness of our derived FFT algorithm an example is presented for $N = 32$ sampled points.

Keywords: Fast fourier transform, Fourier series, Discrete fourier transform, Continuous function.

INTRODUCTION

Over the past several decades, Fourier series and transform have been highly researched by different authors due to their applications in various fields. For instance, Fourier transform is applicable to evolution of neural network and signal processing, resolution of heat equation, filtering technology, modulation and demodulation technology, sampling techniques, digital multimedia visualization systems, performance enhancement and signal processing implementation, examining the ratio of primary energy consumption and the factors that influence carbon emissions in a city and audio signal processing. See for instance the very important works of Smith 2019, Jia et al. 2021,Sun et al. 2019, Hasan et al. 2019, Suganda et al. 2020, Chun et al. 2022 and Nathan 2013. At a glance through the efforts of different authors one sees that the computation of Fourier transform of discretely sampled continuous functions appears to be time and energy wasting due to the complexity of its computational operations. During the early years of the introduction of Fourier series by Jeane Baptiste Fourier, computations of the series were relatively less complicated. Later, Fourier transform was introduced where computations were done at discretely sampled points along the time horizon. Theretofore, computation of discrete Fourier transform (DFT) of continuous functions has been a problem over the years even when amenable to the use of the computer machine. This difficulty in computation results from the large number of complex multiplications followed by complex additions encountered. In order to proffer solution on how to reduce the number of complex computational operations, different mathematicians over the years have pre-occupied themselves with the mathematical formulation or derivations of iterative algorithms generally referred to as Fast Fourier Transform (FFT). For instance, Danielson and Lanczos (1942), derived an algorithm for quick computation of discrete Fourier transform which separated the even part and odd part of the Fourier transform of a continuous function sampled into N points.

The authors considered the representation of the DFT as

$$F(u) = \sum_{x=0}^{N-1} f(x) e^{-\frac{i2\pi ux}{N}} \text{ , u} = 0,1,\ldots,\text{N} - 1 \quad (1)$$

Eq. (1) can be expressed as

$$F(u) = \sum_{x=0}^{N-1} f(x) W_N^{ux} \quad (2)$$

Where we set $\text{W}_\text{N} = \text{e}^{-\text{i}2\pi/\text{N}}$ (3)

So that the sampled points $\text{N} = 2^\text{n}$ for $\text{n} \in \mathbb{Z}_+$. Further the author expressed $\text{N} = 2\text{M}, \Longrightarrow \text{M} = \frac{\text{N}}{2}$ where M is a positive integer substituting for N in Equa.(2) yield,

$$F(u) = \sum_{x=0}^{2M-1} f(x) W_N^{ux}$$

$$= \sum_{x=0}^{M-1} f(2x) W_{2N}^{u(2x)} + \sum_{x=0}^{M-1} f(2x+1) W_{2N}^{u(2x+1)} \quad (4)$$

Here, we have separated the DFT into even and odd indexed terms with W being a complex constant. Cooley and Tukey (1965) formulated a fast Fourier transform Algorithm which was more efficient than the work of Danielson and Lancsoz. The fast Fourier transform algorithm utilized the binary format and produced a better efficiency. Other authors which include, Atonuje and Okonta (2003), Atonuje and Njoseh (2004) and Atonuje (2011) derived other fast Fourier transform (FFT) algorithm formats which utilized other number bases. Although, the above important works presented have contributed to the reduction of complex computational operations of DFT, the derivation of the FFT algorithm had always been based on the choice of the number of sampled points N expressed as indices of some base numbers. The case in which the choice of $N = pq$ where p and q appear to be factors of N is less researched upon or almost missing for existing literature to the best of the authors knowledge. In this article, we present the mathematical derivation of an FFT algorithm where N is composite. Our FFT algorithm is similar to that of the Cooley-Tukey Type algorithm and it is less cumbersome to derive.

METHODOLOGY

Decomposition of number of the sample points N into binary bits of creating summations which are later expanded into block matrix format.

PRELIMINARIES

Consider a continuous function $f(x)$. By discrete Fourier transform DFT, we mean the computation of Fourier transform along sampled points on a continuous function $f(x)$, that is, the values of the FT are recorded at evenly spaced points which are Δx units apart. The function $f(x)$ is now expressed as

$$f(x\_0), f(x\_0 + \Delta x), f(x\_0 + 2\Delta x), f(x\_0 + 3\Delta x), \cdots, f(x\_0 + n\Delta x)$$

where x assumes discrete values $0,1,2,3, \cdots \text{N} - 1$ i.e $\text{f(x} + \text{x}\Delta\text{x)}$.

Here the calculation of Fourier transform is repeated for each value of x. we now define the direct Fourier transform of this discrete function f(x) by the formula

$$F(u) = \sum_{x=0}^{N-1} f(x)e^{-i2\pi ux/N} \quad (5)$$

where $u = 0,1,2,3,\cdots N-1$. Moreover, the corresponding inverse discrete transform from which $f(x)$ is retrieved is given by

$$f(x) = \frac{1}{N}\sum_{u=0}^{N-1} F(U)e^{-i2\pi ux/N} \quad (6)$$

For $x = 0,1,2,\cdots, N-1$.

To be able to see the difficulty associated with the computation of the discrete Fourier transform in Eq. (5), we try a sample easily understandable example.

Example 1

Consider the function $f(x) = x + 1$ sampled at four points whose x co-ordinate value are 0,1,2,3.

Solution

To determine the Discrete Fourier Transform DFT of $f(x)$, we have

$(x_0, x_1, x_2, x_3,) = (0,1,2,3)$, $N = 4$. We need to compute the DFT at these points for $f(x)$. One sees that

$$f(0) = 0 + 1 = 1, \qquad f(1) = 1 + 1 = 2, \qquad f(2) = 2 + 1 = 3, f(3) = 3 + 1 = 4$$

Using Eq. (2.1), we have

$$F(u) = \sum_{x=0}^{N-1} f(x)e^{(-i2\pi ux/N)}$$

For $u = 0,1,2,3$

Hence,

$$F(0) = \sum_{x=0}^{3} f(x)e^{(0)} = \sum_{x=0}^{3} f(x)(1)$$

$$= f(0) + f(1) + f(2) + f(3)$$

$$= 1 + 2 + 3 + 4 = 10$$

$$F(1) = \sum_{x=0}^{3} f(x)e^{\left(-i2\pi(1)x/N\right)}$$

$$= 1e^{0} + 2e^{\left(-\frac{i\pi}{2}\right)} + 3e^{(-\pi i)} + 4e^{\left(-\frac{3\pi i}{2}\right)}$$

$$= 1 - 2i - 3 + 4i = 4i - 2i + 1 - 3 = -2 + 2i$$

$$F(2) = \sum_{x=0}^{3} f(x)e^{(-i4\pi x/N)}$$

$$= 1e^{0} + 2e^{(-\pi i)} + 3e^{(-2\pi i)} + 4e^{(-3\pi i)}$$

$$= 1 - 2 + 3 - 4 = -2 + 0i = -2$$

$$F(3) = \sum_{x=0}^{3} f(x)e^{(-i6\pi x/N)}$$

$$= 1e^{0} + 2e^{\left(-\frac{3\pi i}{2}\right)} + 3e^{(-3\pi i)} + 4e^{\left(-\frac{9\pi i}{2}\right)}$$

$$= 1 + 2i - 3 - 4i = -2 - 2i$$

From the above example, we can see that at the choice of $N = 4$ by direct or machine computation, we have that $N^2 = 4^2 = 16$ complex multiplications and $N(N-1) = 4 \times 3 = 12$ complex additions. Considering the FFT algorithm we see that at $N = 4 = 2^2 \Longrightarrow 2^y, y = 2$. The 16 complex multiplications will reduce to $\frac{Ny}{2} = \frac{4\times 2}{2} = 4$ multiplications and 12 complex additions will reduce to $Ny = 4 \times 2 = 8$ additions. This is quite time saving and justifies the use of the FFT algorithm.

Generally, in computing Discrete Fourier Transform, one is always faced with the question of the number of complex additions and multiplications involved in the calculation of DFT of f(x) sampled at N points. One sees from the example that the evaluation of F(0) requires a total of (2N-1) operations since we have to perform N multiplications i.e $\left(f(x)e \quad \cdots\right)$ and (N-1) additions. Similarly the evaluation if F(1) requires (2N-1) operations and so on until the evaluation of F(u) over all N values of u is complete. This requires a total number of N^2operations i.e multiplications and N(N-1) additions. This means if N=100 the number of complex multiplications will be $N^2 = 100^2 = 10{,}000$ and number of complex additions will be $N(N-1) = 100(100-1) = 9{,}90$. This is discouraging and has been a problem with DFT computation over the years. We present a generalized mathematical derivation of DFT where N has multiple factors.

DERIVATION OF THE GENERALIZED COOLEY-TUKEY TYPE FAST FOURIER TRANSFORM ALGORITHM FOR CONTINUOUS FUNCTION SAMPLED AT $N = h_1 \cdot h_2 \cdots h_r$ with $h_1 \neq h_2 \neq \cdots \neq h_r$

The case considered there is that of the formulation of the Fast Fourier Transform (FFT) algorithm of a continuous function f(x) discretized at $N = h_1 \cdot h_2 \cdots h_r$ points where$h_1 \neq h_2 \neq \cdots \neq h_r$.

Recall that our DFT for N points is represented as

$$F(u) = \sum_{x=0}^{N-1} f_0(x)e^{(-i2\pi ux/N)},$$

$$u = 0,1,2,\cdots,N-1 \qquad (7)$$

Where f(x) are complex values by setting $W = e^{-i2\pi}$ (8)
we obtain or express eq. (7) as

$$F(u) = \sum_{x=0}^{N-1} f(x)W^{ux} \qquad (9)$$

We can see from eq. (9) that the function $f(x)$ is a more of a vector multiplying a matrix W whose entries appear to be (u, x). Clearly W^{ux} and the function $f(x)$ yield a product result in the complex domain. More specifically, for the DFT $F(0)$ we expect N^2 complex multiplication and $N(N-1)$ additions. Our FFT algorithm will aid us to reduce the number of complex operations in the computation of the DFT of $f(x)$.

We now express u and x in decimal bit format as

$$\left.\begin{aligned} u &= u_{m-1}(h_1 \cdot h_2 \cdots h_{m-1}) + u_{m-2}(h_1 \cdot h_2 \cdots h_{m-2}) + u_1 h_1 + u_0 \\ x &= x_{m-1}(h_1 \cdot h_2 \cdots h_m) + x_{m-2}(h_1 \cdot h_2 \cdots h_m) + x_1 h_m + x_0 \end{aligned}\right\} \quad (10)$$

with

$$u_{i-1} = 0,1,2,\cdots,h_{i-1} \qquad 1 \le i \le m$$
$$h_{i-1} = 0,1,2,\cdots,h_{i-1} \qquad 0 \le i \le m-1$$

Using Eq. (10), we now write Eq. (9) as

$$F(u_{m-1}, u_{m-2}, \cdots, u_1 u_0) = \sum_{x_0}\sum_{h_1}\cdots\sum_{x_{m-1}} f_0(h_{m-1} \cdot h_{m-2} \cdots x_0) W^{ux} \quad (11)$$

We note here that $\sum_{h_i}$ is the summation or addition over all $k_i = 0,1,2,\cdots,h_{m-1}$, $0 \le i \le m-1$

It is clear that

$$W^{ux} = W^{u(x_{m-1}[h_2 h_3 \cdots h_{m-1}]+\cdots+x_0)} \quad (12)$$

Considering the first term of the summation

$$W^{ux_{m-1}(h_2 h_3 \cdots h_{m-1})} = W^{[u_{m-1}(h_1 h_3 \cdots h_{m-1})+\cdots+u_0][x_{m-1}(h_2 h_3 \cdots h_m)]}$$
$$= [W^{h_1 h_2 \cdots h_m}]^{(u_{m-1}[h_2 h_3 \cdots h_{m-1}]+u_1)x_{m-1}} \cdots \times W^{u_0 x_{m-1}(h_2 \cdots h_m)} \quad (13)$$

Since $W^{h_1 h_2 \cdots h_m} = W^N = 1$, Eq. (13) can be expressed as

$$W^{ux_{m-1}(h_1 h_2 \cdots h_m)} = W^{ux_{m-1}(h_2 h_3 \cdots h_m)} \quad (14)$$

Eq. (12) can now be expressed as

$$W^{ux} = W^{ux_{m-1}(h_1 h_2 \cdots h_m)} \boldsymbol{W^{ux_{m-1}[h_2 h_3 \cdots x_m]}} \quad (15)$$

Eq. (11) can again be expressed as

$$F(u_{m-1}, u_{m-2}, \cdots, u_1 u_0) = \sum_{x_0}\sum_{h_1}\cdots\sum_{x_{m-1}} f(u_0 \cdot x_{m-2} \cdots x_0) \times W^{u[x_{m-2}(h_2 \cdots h_m)+\cdots+x_0]} \quad (16)$$

Following the same pattern which led to Eq. (15) we obtain

$$W^{ux_{m-2}(h_3h_4\cdots h_m)} = W^{(u_1h_1+u_0)x_{m-2}(h_1h_4\cdots h_m)} \quad (17)$$

The identity in Eq. (17) permits the inner of Eq. (16) to be expressed as

$$f_2(u_1,u_0,\cdots,x_{m-3},\cdots,x_0) = \sum_{x_{m-2}} f_1(u_0\cdot x_{m-2}\cdots x_0) \times W^{(u_1h_1+u_0)x_{m-2}(h_1h_4\cdots h_m)} \quad (18)$$

Eq. (16) can now be written as

$$F(u_{m-1},u_{m-2},\cdots,u_1,u_0) = \sum_{x_0}\sum_{h_1}\cdots\sum_{x_{m-2}} f_2(u_1,u_0,\cdots,x_{m-3},\cdots,x_0) \times W^{u[x_{m-1}(h_4h_5\cdots h_m)+\cdots+x_0]} \quad (19)$$

By continuously reducing Eq. (19) we obtain a set of recursive equations of the form

$$f_1(u_0,u_1,\cdots,x_{m-1},\cdots,x_0) = \sum_{x_{m-1}} f_{i-1}(u_0,u_1,\cdots u_{i-2},x_{m-2}\cdots x_0) \times W^{[u_{i-1}(h_1h_2\cdots h_{i-1})+\cdots+u_0)x_{m-1}(h_{i-1}\cdots h_m)} \quad (20)$$

Eq. (20) represents an extension of the original Cooley-Tukey FFT algorithm for the general case whereby a continuous function $f(x)$ is sample d at N points where $N = h_1\cdot h_2\cdots h_m$and the h_i are the factors of N. the expression is valid so long as we define $(p_{i+1}\cdots p_m) = I$ for $I > m-i$ and $x-1=0$. The concluding result of the FFT algorithm is given as

$$F(u_{m-1},\cdots,u_0) = f_m(u_0,\cdots,u_{m-1}) \quad (21)$$

APPLICATON OF DISCRETE FOURIER TRANSFORM THE CHOICE OF COMPOSITE NUMBER OF SAMPLED POINTS

In this section, we shall sample the continuous function at N points where N is highly composite. To this end, we choose $N = 32$ and factor it as $N = 32 = 8\times 4$ where $p = 8$ and$q = 4$ i.e $p \neq q$.

The procedure we shall adopt involves the derivation of as many arrays as possible with a base 4 algorithm and then develop a binary array.

Let us first run a substitution as $p = 8$ and $q = 4$ and express base "8+4" into the usual binary equivalent. So that we obtain from the DFT

$$F(u) = \sum_{x=0}^{N-1} f_0(x)e^{(-i2\pi x/N)} \quad (22)$$

where $f(x)$ are complex values. Using the substitution

$$W = e^{(-i2\pi/N)} \tag{23}$$

We can now express Eq. (22) as

$$F(u) = \sum_{x=0}^{N-1} f(x)W^{ux} \tag{24}$$

The exponents u and x in Eq. (24) now become

$$\left.\begin{aligned} u &= 8u_1 + u_0;\ u_0 = 0,1,2,3,4,5,6,7\ \ u_1 = 0,1,2,3 \\ x &= 4x_1 + x_0;\ x_0 = 0,1,2,3\ \ x_1 = 0,1,2,3,4,5,6,7 \end{aligned}\right\} \tag{25}$$

We can now see from the DFT

$$F(u) = \sum_{x=0}^{N-1} f(x)W^{ux}, \quad u = 0,1,2,\cdots,N-1 \tag{26}$$

Where we have the set $W = e^{(-i2\pi/N)}$, the form, the form (26) is now expressed as

$$F(u_1,u_0) = \sum_{x_0=0}^{3}\sum_{x_1=0}^{7} f_0(x_1,x_0)W^{(8u_1+u_0)(4x_1+x_0)} \tag{27}$$

$$W^{(8u_1+u_0)(4x_1+x_0)} = W^{(8u_1+u_0)4x_1}W^{(8u_1+u_0)x_0}$$

$$= [W^{32u_1x_1}]W^{4u_0x_1}W^{(8u_1+u_0)x_0}$$

$$= W^{4u_0x_1}W^{(8u_1+u_0)x_0}$$

Note the term in the bracket equal to unity

$$W^{32u_1x_1} = [W^{32}]^{u_1x_1}$$

Recall from (23)

$$= \left[e^{-\left[\frac{2\pi}{32}\right]32}\right]^{u_1x_1}$$

$$= \left[e^{-i2\pi}\right]^{u_1x_1}$$

$$= 1$$

Thus we have

$$F(u_1,u_0) = \sum_{x_0=0}^{3}\left[\sum_{x_1=0}^{7} f_0(x_1,x_0)W^{4u_0x_1}\right]W^{(8u_1+u_0)x_0} \tag{28}$$

From (28) we have the inner sum

$$f_1(u_0, x_0) = \sum_{x_{1=0}}^{7} f_0(x_1, x_0) W^{4u_0x_1} \quad (29)$$

The outer sum

$$f_2(u_0, u_1) = \sum_{x_{1=0}}^{3} f_1(u_0, x_0) W^{(8u_1+u_0)x_0} \quad (30)$$

And the unscrambling is accomplished according to the relationship

$$F(u_1, u_0) = f_2(u_0, u_1) \quad (31)$$

From (29) we have that

$$\begin{aligned} f_1(0,0) = {} & f_0(0,0)W^{4(0)(0)} + f_0(0,1)W^{4(0)(0)} + f_0(0,2)W^{4(0)(0)} + f_0(0,3)W^{4(0)(0)} \\ & + f_0(1,0)W^{4(0)(1)} + f_0(1,1)W^{4(0)(1)} + f_0(1,2)W^{4(0)(1)} + f_0(1,2)W^{4(0)(1)} \\ & + f_0(1,3)W^{4(0)(1)} + f_0(2,0)W^{4(0)(2)} + f_0(2,1)W^{4(0)(2)} + f_0(2,2)W^{4(0)(2)} \\ & + f_0(2,3)W^{4(0)(2)} + f_0(3,0)W^{4(0)(3)} + f_0(3,1)W^{4(0)(3)} + f_0(3,2)W^{4(0)(3)} \\ & + f_0(3,3)W^{4(0)(3)} + f_0(4,0)W^{4(0)(4)} + f_0(4,1)W^{4(0)(4)} + f_0(4,2)W^{4(0)(4)} \\ & + f_0(4,3)W^{4(0)(4)} + f_0(5,0)W^{4(0)(5)} + f_0(5,1)W^{4(0)(5)} + f_0(5,2)W^{4(0)(5)} \\ & + f_0(5,3)W^{4(0)(5)} + f_0(6,0)W^{4(0)(6)} + f_0(6,1)W^{4(0)(6)} + f_0(6,2)W^{4(0)(6)} \\ & + f_0(6,3)W^{4(0)(6)} + f_0(7,0)W^{4(0)(7)} + f_0(7,1)W^{4(0)(7)} + f_0(7,2)W^{4(0)(7)} \\ & + f_0(7,3)W^{4(0)(7)} \end{aligned}$$

$$\vdots$$

$$\begin{aligned} f_1(7,3) = {} & f_0(0,0)W^{4(7)(0)} + f_0(0,1)W^{4(7)(0)} + f_0(0,2)W^{4(7)(0)} + f_0(0,3)W^{4(7)(0)} \\ & + f_0(1,0)W^{4(7)(1)} + f_0(1,1)W^{4(7)(1)} + f_0(1,2)W^{4(7)(1)} + f_0(1,2)W^{4(7)(1)} \\ & + f_0(1,3)W^{4(7)(1)} + f_0(2,0)W^{4(7)(2)} + f_0(2,1)W^{4(7)(2)} + f_0(2,2)W^{4(7)(2)} \\ & + f_0(2,3)W^{4(7)(2)} + f_0(3,0)W^{4(7)(3)} + f_0(3,1)W^{4(7)(3)} + f_0(3,2)W^{4(7)(3)} \\ & + f_0(3,3)W^{4(7)(3)} + f_0(4,0)W^{4(7)(4)} + f_0(4,1)W^{4(7)(4)} + f_0(4,2)W^{4(7)(4)} \\ & + f_0(4,3)W^{4(7)(4)} + f_0(5,0)W^{4(7)(5)} + f_0(5,1)W^{4(7)(5)} + f_0(5,2)W^{4(7)(5)} \\ & + f_0(5,3)W^{4(7)(5)} + f_0(6,0)W^{4(7)(6)} + f_0(6,1)W^{4(7)(6)} + f_0(6,2)W^{4(7)(6)} \\ & + f_0(6,3)W^{4(7)(6)} + f_0(7,0)W^{4(0)(7)} + f_0(7,1)W^{4(7)(7)} + f_0(7,2)W^{4(7)(7)} \\ & + f_0(7,3)W^{4(7)(7)} \end{aligned}$$

From $f_1(0,0)$ we see that $x_0 = 0$, so we can say that at any point $x_0 \neq 0$ from $f_1(0,0)$ then we equate it to zero and it will continue in that order from the respective values from $f_1(0,0)$ to $f_1(7,3)$.

$$
\begin{bmatrix} f_1(0,0) \\ f_1(0,1) \\ f_1(0,2) \\ f_1(0,3) \\ f_1(1,0) \\ f_1(1,1) \\ f_1(1,2) \\ f_1(1,3) \\ f_1(2,0) \\ f_1(2,1) \\ f_1(2,2) \\ f_1(2,3) \\ f_1(3,0) \\ f_1(3,1) \\ f_1(3,2) \\ f_1(3,3) \\ f_1(4,0) \\ f_1(4,1) \\ f_1(4,2) \\ f_1(4,3) \\ f_1(5,0) \\ f_1(5,1) \\ f_1(5,2) \\ f_1(5,3) \\ f_1(6,0) \\ f_1(6,1) \\ f_1(6,2) \\ f_1(6,3) \\ f_1(7,0) \\ f_1(7,1) \\ f_1(7,2) \\ f_1(7,3) \end{bmatrix}
=
\begin{bmatrix} W^0 & 0 & 0 & 0 & W^0 & 0 & 0 & 0 & W^0 & 0 & 0 & 0 & W^0 & 0 & 0 & 0 \\ 0 & W^0 & 0 & 0 & 0 & W^0 & 0 & 0 & 0 & W^0 & 0 & 0 & 0 & W^0 & 0 & 0 \\ 0 & 0 & W^0 & 0 & 0 & 0 & W^0 & 0 & 0 & 0 & W^0 & 0 & 0 & 0 & W^0 & 0 \\ 0 & 0 & 0 & W^0 & 0 & 0 & 0 & W^0 & 0 & 0 & 0 & W^0 & 0 & 0 & 0 & W^0 \\ W^0 & 0 & 0 & 0 & W^4 & 0 & 0 & 0 & W^8 & 0 & 0 & 0 & W^{12} & 0 & 0 & 0 \\ 0 & W^0 & 0 & 0 & 0 & W^4 & 0 & 0 & 0 & W^8 & 0 & 0 & 0 & W^{12} & 0 & 0 \\ 0 & 0 & W^0 & 0 & 0 & 0 & W^4 & 0 & 0 & 0 & W^8 & 0 & 0 & 0 & W^{12} & 0 \\ 0 & 0 & 0 & W^0 & 0 & 0 & 0 & W^4 & 0 & 0 & 0 & W^8 & 0 & 0 & 0 & W^{12} \\ W^0 & 0 & 0 & 0 & W^8 & 0 & 0 & 0 & W^{16} & 0 & 0 & 0 & W^{24} & 0 & 0 & 0 \\ 0 & W^0 & 0 & 0 & 0 & W^8 & 0 & 0 & 0 & W^{16} & 0 & 0 & 0 & W^{24} & 0 & 0 \\ 0 & 0 & W^0 & 0 & 0 & 0 & W^8 & 0 & 0 & 0 & W^{16} & 0 & 0 & 0 & W^{24} & 0 \\ 0 & 0 & 0 & W^0 & 0 & 0 & 0 & W^8 & 0 & 0 & 0 & W^{16} & 0 & 0 & 0 & W^{24} \\ W^0 & 0 & 0 & 0 & W^{12} & 0 & 0 & 0 & W^{24} & 0 & 0 & 0 & W^{36} & 0 & 0 & 0 \\ 0 & W^0 & 0 & 0 & 0 & W^{12} & 0 & 0 & 0 & W^{24} & 0 & 0 & 0 & W^{36} & 0 & 0 \\ 0 & 0 & W^0 & 0 & 0 & 0 & W^{12} & 0 & 0 & 0 & W^{24} & 0 & 0 & 0 & W^{36} & 0 \\ 0 & 0 & 0 & W^0 & 0 & 0 & 0 & W^{12} & 0 & 0 & 0 & W^{24} & 0 & 0 & 0 & W^{36} \\ W^0 & 0 & 0 & 0 & W^{16} & 0 & 0 & 0 & W^{32} & 0 & 0 & 0 & W^{48} & 0 & 0 & 0 \\ 0 & W^0 & 0 & 0 & 0 & W^{16} & 0 & 0 & 0 & W^{32} & 0 & 0 & 0 & W^{48} & 0 & 0 \\ 0 & 0 & W^0 & 0 & 0 & 0 & W^{16} & 0 & 0 & 0 & W^{32} & 0 & 0 & 0 & W^{48} & 0 \\ 0 & 0 & 0 & W^0 & 0 & 0 & 0 & W^{16} & 0 & 0 & 0 & W^{32} & 0 & 0 & 0 & W^{48} \\ W^0 & 0 & 0 & 0 & W^{20} & 0 & 0 & 0 & W^{40} & 0 & 0 & 0 & W^{60} & 0 & 0 & 0 \\ 0 & W^0 & 0 & 0 & 0 & W^{20} & 0 & 0 & 0 & W^{40} & 0 & 0 & 0 & W^{60} & 0 & 0 \\ 0 & 0 & W^0 & 0 & 0 & 0 & W^{20} & 0 & 0 & 0 & W^{40} & 0 & 0 & 0 & W^{60} & 0 \\ 0 & 0 & 0 & W^0 & 0 & 0 & 0 & W^{20} & 0 & 0 & 0 & W^{40} & 0 & 0 & 0 & W^{60} \\ W^0 & 0 & 0 & 0 & W^{24} & 0 & 0 & 0 & W^{48} & 0 & 0 & 0 & W^{72} & 0 & 0 & 0 \\ 0 & W^0 & 0 & 0 & 0 & W^{24} & 0 & 0 & 0 & W^{48} & 0 & 0 & 0 & W^{72} & 0 & 0 \\ 0 & 0 & W^0 & 0 & 0 & 0 & W^{24} & 0 & 0 & 0 & W^{48} & 0 & 0 & 0 & W^{72} & 0 \\ 0 & 0 & 0 & W^0 & 0 & 0 & 0 & W^{24} & 0 & 0 & 0 & W^{48} & 0 & 0 & 0 & W^{72} \\ W^0 & 0 & 0 & 0 & W^{28} & 0 & 0 & 0 & W^{56} & 0 & 0 & 0 & W^{84} & 0 & 0 & 0 \\ 0 & W^0 & 0 & 0 & 0 & W^{28} & 0 & 0 & 0 & W^{56} & 0 & 0 & 0 & W^{84} & 0 & 0 \\ 0 & 0 & W^0 & 0 & 0 & 0 & W^{28} & 0 & 0 & 0 & W^{56} & 0 & 0 & 0 & W^{84} & 0 \\ 0 & 0 & 0 & W^0 & 0 & 0 & 0 & W^{28} & 0 & 0 & 0 & W^{56} & 0 & 0 & 0 & W^{84} \end{bmatrix}
\begin{bmatrix} f_1(0,0) \\ f_1(0,1) \\ f_1(0,2) \\ f_1(0,3) \\ f_1(1,0) \\ f_1(1,1) \\ f_1(1,2) \\ f_1(1,3) \\ f_1(2,0) \\ f_1(2,1) \\ f_1(2,2) \\ f_1(2,3) \\ f_1(3,0) \\ f_1(3,1) \\ f_1(3,2) \\ f_1(3,3) \\ f_1(4,0) \\ f_1(4,1) \\ f_1(4,2) \\ f_1(4,3) \\ f_1(5,0) \\ f_1(5,1) \\ f_1(5,2) \\ f_1(5,3) \\ f_1(6,0) \\ f_1(6,1) \\ f_1(6,2) \\ f_1(6,3) \\ f_1(7,0) \\ f_1(7,1) \\ f_1(7,2) \\ f_1(7,3) \end{bmatrix}
\quad (32)
$$

Where $W = e^{-\frac{2\pi}{N}}$, thus

$$W^0 = e^{\left(-\frac{2\pi}{N}\right)(0)}$$

$$W^0 = e^0$$

$$W^0 = 1$$

The matrix below is gotten from the fact that $W^{ux} = W^{uxmod(N)}$

$$
\begin{bmatrix} f_1(0,0) \\ f_1(0,1) \\ f_1(0,2) \\ f_1(0,3) \\ f_1(1,0) \\ f_1(1,1) \\ f_1(1,2) \\ f_1(1,3) \\ f_1(2,0) \\ f_1(2,1) \\ f_1(2,2) \\ f_1(2,3) \\ f_1(3,0) \\ f_1(3,1) \\ f_1(3,2) \\ f_1(3,3) \\ f_1(4,0) \\ f_1(4,1) \\ f_1(4,2) \\ f_1(4,3) \\ f_1(5,0) \\ f_1(5,1) \\ f_1(5,2) \\ f_1(5,3) \\ f_1(6,0) \\ f_1(6,1) \\ f_1(6,2) \\ f_1(6,3) \\ f_1(7,0) \\ f_1(7,1) \\ f_1(7,2) \\ f_1(7,3) \end{bmatrix}
=
\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & W^{16} & 0 & 0 & 0 & 0 & W^{20} & 0 & 0 & 0 & 0 & W^{28} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & W^{16} & 0 & 0 & 0 & 0 & W^{20} & 0 & 0 & 0 & 0 & W^{28} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & W^{16} & 0 & 0 & 0 & 0 & W^{20} & 0 & 0 & 0 & 0 & W^{28} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & W^{16} & 0 & 0 & 0 & 0 & W^{20} & 0 & 0 & 0 & 0 & W^{28} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & W^{16} & 0 & 0 & 0 & 0 & W^{20} & 0 & 0 & 0 & 0 & W^{28} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & W^{16} & 0 & 0 & 0 & 0 & W^{20} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & W^{16} & 0 & 0 & 0 & 0 & W^{20} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & W^{16} & 0 & 0 & 0 & 0 & W^{20} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & W^{16} & 0 & 0 & 0 & 0 & W^{20} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & W^{16} & 0 & 0 & 0 & 0 & W^{20} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & W^{16} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & W^{16} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & W^{16} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & W^{16} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & W^{16} \end{bmatrix}
\begin{bmatrix} f_1(0,0) \\ f_1(0,1) \\ f_1(0,2) \\ f_1(0,3) \\ f_1(1,0) \\ f_1(1,1) \\ f_1(1,2) \\ f_1(1,3) \\ f_1(2,0) \\ f_1(2,1) \\ f_1(2,2) \\ f_1(2,3) \\ f_1(3,0) \\ f_1(3,1) \\ f_1(3,2) \\ f_1(3,3) \\ f_1(4,0) \\ f_1(4,1) \\ f_1(4,2) \\ f_1(4,3) \\ f_1(5,0) \\ f_1(5,1) \\ f_1(5,2) \\ f_1(5,3) \\ f_1(6,0) \\ f_1(6,1) \\ f_1(6,2) \\ f_1(6,3) \\ f_1(7,0) \\ f_1(7,1) \\ f_1(7,2) \\ f_1(7,3) \end{bmatrix}
\quad (33)
$$

From the outer sum (3.9) we have that

$$\begin{aligned}f_2(0,0) = {} & f_1(0,0)W^{(8\times0+0)0} + f_1(0,1)W^{(8\times0+0)1} + f_1(0,2)W^{(8\times0+0)2} + f_1(0,3)W^{(8\times0+0)3} \\ & + f_1(1,0)W^{(8\times0+0)0} + f_1(1,1)W^{(8\times0+0)1} + f_1(1,2)W^{(8\times0+0)2} \\ & + f_1(1,3)W^{(8\times0+0)3} + f_1(2,0)W^{(8\times0+0)0} + f_1(2,1)W^{(8\times0+0)1} \\ & + f_1(2,2)W^{(8\times0+0)2} + f_1(2,3)W^{(8\times0+0)3} + f_1(3,0)W^{(8\times0+0)0} \\ & + f_1(3,1)W^{(8\times0+0)1} + f_1(3,2)W^{(8\times0+0)2} + f_1(3,3)W^{(8\times0+0)3} \\ & + f_1(4,0)W^{(8\times0+0)0} + f_1(4,1)W^{(8\times0+0)1} + f_1(4,2)W^{(8\times0+0)2} \\ & + f_1(4,3)W^{(8\times0+0)3} + f_1(5,0)W^{(8\times0+0)0} + f_1(5,1)W^{(8\times0+0)1} \\ & + f_1(5,2)W^{(8\times0+0)2} + f_1(5,3)W^{(8\times0+0)3} + f_1(6,0)W^{(8\times0+0)0} \\ & + f_1(6,1)W^{(8\times0+0)1} + f_1(6,2)W^{(8\times0+0)2} + f_1(6,3)W^{(8\times0+0)3} \\ & + f_1(7,0)W^{(8\times0+0)0} + f_1(7,1)W^{(8\times0+0)1} + f_1(7,2)W^{(8\times0+0)2} \\ & + f_1(7,3)W^{(8\times0+0)3}\end{aligned}$$

$$\vdots$$

$$\begin{aligned}f_2(7,3) = {} & f_1(0,0)W^{(8\times3+7)0} + f_1(0,1)W^{(8\times3+7)1} + f_1(0,2)W^{(8\times3+7)2} + f_1(0,3)W^{(8\times3+7)3} \\ & + f_1(1,0)W^{(8\times3+7)0} + f_1(1,1)W^{(8\times3+7)1} + f_1(1,2)W^{(8\times3+7)2} \\ & + f_1(1,3)W^{(8\times3+7)3} + f_1(2,0)W^{(8\times3+7)0} + f_1(2,1)W^{(8\times3+7)1} \\ & + f_1(2,2)W^{(8\times3+7)2} + f_1(2,3)W^{(8\times3+7)3} + f_1(3,0)W^{(8\times3+7)0} \\ & + f_1(3,1)W^{(8\times3+7)1} + f_1(3,2)W^{(8\times3+7)2} + f_1(3,3)W^{(8\times3+7)3} \\ & + f_1(4,0)W^{(8\times3+7)0} + f_1(4,1)W^{(8\times3+7)1} + f_1(4,2)W^{(8\times3+7)2} \\ & + f_1(4,3)W^{(8\times3+7)3} + f_1(5,0)W^{(8\times3+7)0} + f_1(5,1)W^{(8\times3+7)1} \\ & + f_1(5,2)W^{(8\times3+7)2} + f_1(5,3)W^{(8\times3+7)3} + f_1(6,0)W^{(8\times3+7)0} \\ & + f_1(6,1)W^{(8\times3+7)1} + f_1(6,2)W^{(8\times3+7)2} + f_1(6,3)W^{(8\times3+7)3} \\ & + f_1(7,0)W^{(8\times3+7)0} + f_1(7,1)W^{(8\times3+7)1} + f_1(7,2)W^{(8\times3+7)2} \\ & + f_1(7,3)W^{(8\times3+7)3}\end{aligned}$$

From $f_2(0,0)$ we see that $u_0 = 0$, so we can say that at any point $u_0 \neq 0$ from $f_2(0,0)$ then we equate it to zero and it will continue in that order from the given respective values from $f_2(0,0)$ to $f_2(7,3)$.

$$
\begin{bmatrix} f_2(0,0) \\ f_2(0,1) \\ f_2(0,2) \\ f_2(0,3) \\ f_2(1,0) \\ f_2(1,1) \\ f_2(1,2) \\ f_2(1,3) \\ f_2(2,0) \\ f_2(2,1) \\ f_2(2,2) \\ f_2(2,3) \\ f_2(3,0) \\ f_2(3,1) \\ f_2(3,2) \\ f_2(3,3) \\ f_2(4,0) \\ f_2(4,1) \\ f_2(4,2) \\ f_2(4,3) \\ f_2(5,0) \\ f_2(5,1) \\ f_2(5,2) \\ f_2(5,3) \\ f_2(6,0) \\ f_2(6,1) \\ f_2(6,2) \\ f_2(6,3) \\ f_2(7,0) \\ f_2(7,1) \\ f_2(7,2) \\ f_2(7,3) \end{bmatrix}
=
\begin{bmatrix}
W^0 & W^0 & W^0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
W^0 & W^8 & W^{16} & W^{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
W^0 & W^{16} & W^{32} & W^{48} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
W^0 & W^{24} & W^{48} & W^{72} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & W^0 & W^1 & W^2 & W^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & W^0 & W^9 & W^{18} & W^{27} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & W^0 & W^{17} & W^{34} & W^{51} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & W^0 & W^{25} & W^{50} & W^{75} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & W^0 & W^2 & W^4 & W^6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & W^0 & W^{10} & W^{20} & W^{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & W^0 & W^{18} & W^{36} & W^{54} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & W^0 & W^{26} & W^{52} & W^{78} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & W^0 & W^3 & W^6 & W^9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & W^0 & W^{11} & W^{22} & W^{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & W^0 & W^{19} & W^{38} & W^{57} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & W^0 & W^{27} & W^{54} & W^{81} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & W^0 & W^4 & W^8 & W^{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & W^0 & W^{12} & W^{24} & W^{36} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & W^0 & W^{20} & W^{40} & W^{60} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & W^0 & W^{28} & W^{56} & W^{84} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & W^0 & W^5 & W^{10} & W^{15} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & W^0 & W^{13} & W^{26} & W^{36} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & W^0 & W^{21} & W^{42} & W^{63} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & W^0 & W^{29} & W^{58} & W^{87} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & W^0 & W^6 & W^{12} & W^{18} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & W^0 & W^{14} & W^{28} & W^{42} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & W^0 & W^{22} & W^{44} & W^{66} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & W^0 & W^{20} & W^{60} & W^{90} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & W^0 & W^7 & W^{14} & W^{21} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & W^0 & W^{15} & W^{30} & W^{45} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & W^0 & W^{23} & W^{46} & W^{69} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & W^0 & W^{31} & W^{62} & W^{186}
\end{bmatrix}
\begin{bmatrix} f_2(0,0) \\ f_2(0,1) \\ f_2(0,2) \\ f_2(0,3) \\ f_2(1,0) \\ f_2(1,1) \\ f_2(1,2) \\ f_2(1,3) \\ f_2(2,0) \\ f_2(2,1) \\ f_2(2,2) \\ f_2(2,3) \\ f_2(3,0) \\ f_2(3,1) \\ f_2(3,2) \\ f_2(3,3) \\ f_2(4,0) \\ f_2(4,1) \\ f_2(4,2) \\ f_2(4,3) \\ f_2(5,0) \\ f_2(5,1) \\ f_2(5,2) \\ f_2(5,3) \\ f_2(6,0) \\ f_2(6,1) \\ f_2(6,2) \\ f_2(6,3) \\ f_2(7,0) \\ f_2(7,1) \\ f_2(7,2) \\ f_2(7,3) \end{bmatrix}
\quad (3.4)
$$

$$W^0 \ is \ not \ reduced \ to \ unity \ in \ other \ to \ have \ a \ generalized \ result$$

Where $W = e^{-\frac{2\pi}{N}}$, thus

$$W^0 = e^{\left(-\frac{2\pi}{N}\right)(0)}$$

$$W^0 = e^0$$

$$W^0 = 1$$

The matrix below is gotten from the fact that $W^{ux} = W^{uxmod(N)}$

$$\begin{bmatrix} f_2(0,0)\\ f_2(0,1)\\ f_2(0,2)\\ f_2(0,3)\\ f_2(1,0)\\ f_2(1,1)\\ f_2(1,2)\\ f_2(1,3)\\ f_2(2,0)\\ f_2(2,1)\\ f_2(2,2)\\ f_2(2,3)\\ f_2(3,0)\\ f_2(3,1)\\ f_2(3,2)\\ f_2(3,3)\\ f_2(4,0)\\ f_2(4,1)\\ f_2(4,2)\\ f_2(4,3)\\ f_2(5,0)\\ f_2(5,1)\\ f_2(5,2)\\ f_2(5,3)\\ f_2(6,0)\\ f_2(6,1)\\ f_2(6,2)\\ f_2(6,3)\\ f_2(7,0)\\ f_2(7,1)\\ f_2(7,2)\\ f_2(7,3) \end{bmatrix} = \begin{bmatrix}
1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&W^{8}&W^{16}&W^{24}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&W^{16}&W^{0}&W^{16}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&W^{24}&W^{16}&W^{8}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&W^{0}&W^{1}&W^{2}&W^{3}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&W^{0}&W^{9}&W^{18}&W^{27}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&W^{0}&W^{17}&W^{2}&W^{13}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&W^{0}&W^{25}&W^{12}&W^{11}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&W^{0}&W^{2}&W^{4}&W^{6}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&W^{0}&W^{10}&W^{20}&W^{14}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&W^{0}&W^{18}&W^{36}&W^{10}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&W^{0}&W^{26}&W^{12}&W^{14}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&W^{0}&W^{3}&W^{6}&W^{9}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&W^{0}&W^{11}&W^{22}&W^{1}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&W^{0}&W^{19}&W^{6}&W^{7}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&W^{0}&W^{27}&W^{10}&W^{17}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&W^{0}&W^{4}&W^{8}&W^{12}&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&W^{0}&W^{12}&W^{24}&W^{4}&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&W^{0}&W^{20}&W^{4}&W^{60}&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&W^{0}&W^{28}&W^{24}&W^{28}&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&W^{0}&W^{5}&W^{10}&W^{15}&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&W^{0}&W^{13}&W^{26}&W^{4}&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&W^{0}&W^{21}&W^{10}&W^{31}&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&W^{0}&W^{29}&W^{27}&W^{23}&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&W^{0}&W^{6}&W^{12}&W^{18}&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&W^{0}&W^{14}&W^{28}&W^{10}&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&W^{0}&W^{22}&W^{12}&W^{2}&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&W^{0}&W^{20}&W^{4}&W^{26}&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&W^{0}&W^{7}&W^{14}&W^{21}\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&W^{0}&W^{15}&W^{30}&W^{13}\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&W^{0}&W^{23}&W^{14}&W^{5}\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&W^{0}&W^{31}&W^{30}&W^{26}
\end{bmatrix} \begin{bmatrix} f_2(0,0)\\ f_2(0,1)\\ f_2(0,2)\\ f_2(0,3)\\ f_2(1,0)\\ f_2(1,1)\\ f_2(1,2)\\ f_2(1,3)\\ f_2(2,0)\\ f_2(2,1)\\ f_2(2,2)\\ f_2(2,3)\\ f_2(3,0)\\ f_2(3,1)\\ f_2(3,2)\\ f_2(3,3)\\ f_2(4,0)\\ f_2(4,1)\\ f_2(4,2)\\ f_2(4,3)\\ f_2(5,0)\\ f_2(5,1)\\ f_2(5,2)\\ f_2(5,3)\\ f_2(6,0)\\ f_2(6,1)\\ f_2(6,2)\\ f_2(6,3)\\ f_2(7,0)\\ f_2(7,1)\\ f_2(7,2)\\ f_2(7,3) \end{bmatrix}$$

$$\left.\begin{aligned} f_1(u_0, x_0) &= \sum_{x_1=0}^{7} f_0(x_1, x_0) W^{4u_0 x_1} \\ f_2(u_0, u_1) &= \sum_{x_0=0}^{3} f_1(u_0, x_0) W^{(8u_1+u_0)x_0} \\ F(u_1, u_0) &= f_2(u_0, u_1) \end{aligned}\right\} \quad (36)$$

Since the second equation is formed in terms of the first, we refer to these equations as recursive. The discrete Fourier Transform (24) is algebraically split into factored matrices with zeros as a result of the FFT technique as it is presented above.

CONCLUSION

By ordinary calculation of DFT we have that the computation of $f_1(u_0, x_0), f_2(u_0, u_1)$ would require N^2 complex multiplications and $N(N-1)$ complex additions. The FFT algorithm derived helped to introduce zeros into the square diagonal matrices as well as zeros elsewhere. This resulted in the reduction of the complex multiplications from $N^2 = 32^2 = 1024$ to $\frac{Ny}{2} = \frac{32\times5}{2} = 80$ and complex additions from $N(N-1) = 32(32-1) = 992$ to $Ny = 32 \times 5 = 160$ additions (where y is an integer to which the selected base number is raised.

CONFLICT OF INTEREST

The authors have not declared any conflict of interest.

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