APPROXIMATE SOLUTIONS OF FRACTIONAL FREDHOLM-TYPE INTEGRO-DIFFERENTIAL EQUATIONS WITH CERTAIN ORTHOGONAL POLYNOMIAL

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Abstract

This study focused on obtaining the approximate solution of the fractional integro-differential equation using the orthogonal collocation method (OCM) with a certain orthogonal polynomial, constructed in the interval of $[0,1]$ (where 0 and 1 are in radian) with respect to the weight function $w(x) = cosx$. The orthogonal collocation method (OCM) was adopted as a numerical solver with the generated orthogonal polynomial as basis functions. Numerical experiments with MAPLE 18 showed that the method exhibits rapid convergence. Also, stability analysis of the method showed that the method is stable with minimal error. Hence, the newly constructed orthogonal polynomial was satisfactorily employed to obtain the approximate solution of fractional integro-differential equations with better error estimates when compared with other methods in the literature.

Keywords: Orthogonal polynomial, Collocation method, Integro-differential equation Fractional derivatives, Approximate solution.

1. Introduction

The fractional integro-differential equation is known to model many physical phenomena such as earthquakes, traffic flow, and viscoelastic material properties (Oyedepo, *et al*, 2016). Most of these problems cannot be solved through analytical methods due to some requirements in perturbation, linearization, and quasi-linearization. Consequently, numerical methods are preferred to seek approximate solutions to these problems.

Many scholars have developed different numerical methods for the solution of fractional integro-differential equations. For instance, the collocation method (Bhrawy and Alghamdi, 2012) was used for solving the Langevin fractional equation of nonlinear order in two different intervals. The authors equally solved the fractional Fredholm integro-differential equation. In the same vain, Bhrawy and Alofi (2013), Doha *et al*. (2011), and Irandoust-Pakchin, *et al* (2013) applied Chebyshev polynomials as basis functions for the solution of differential equations involving multiterm fractional order and nonlinear Fredholm and Volterra integro-differential equations. Irandoust-Pacchin and Abdi-Mazraeh (2013) solved fractional integro-differential equations with nonlocal boundary conditions using the variational iteration method (VIM). The Adomian decomposition method (ADM) was used by the authors; Saha (2009) and Mittal and Nigam (2008) to solve both fractional integro-differential and diffusion equations.The system of linear Fredholm fractional integro-differential of fractional order was solved by Saeedi and Samimi (2012) using the Homotopy perturbation method.

The use of orthogonal polynomials as basis function for the approximate solutions of differential equations started in the 1930s (Brunner, 2014). Over the years, different scholars have been on the hunt for orthogonal polynomials as basis functions for the solution of many problems in science and engineering. For instance, Olagunju and Joseph (2013) solved the fractional diffusion equation by adopting the first kind Chebyshev orthogonal polynomials as basis functions. Zhang and Liu (2006) applied the Laguerre polynomials as basis functions for the solution of nonlinear boundary value problems in a spectral collocation method. In like manner, the Mamadu-Njoseh polynomials have been adopted as basis functions to seek the solutions to many problems in mathematics (Njoseh and Mamadu, 2016; Mamadu and Njoseh, 2016a; Mamadu and Njoseh, 2016b).

The motivation behind this paper is the construction of certain orthogonal polynomial with weight function, $w(x) = cosx,\ x \in [0,1]$, (where end points are in radian) with the view of solving the fractional integro-differential equation of Fredholm kind. The orthogonal collocation method shall be the solver through which the orthogonal polynomial will be launched as basis functions.

2. Preliminaries and Definitions

Definitions:

2.1: Caputo (1967) defines the Caputo fractional derivatives for a function $g(t)$ as

$$ {}_a^cD_t^\alpha g(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_a^t (t-s)^{m-\alpha-1}[D^m g(s)]ds, & where\ m-1<\alpha<m \\ \frac{d^m}{dt^m} g(t), & where\ \alpha = m \end{cases}, \qquad (2.1)$$

2.2: The Riemann-Liouville fractional derivative of order α defined in $m-1<\alpha<m$ is given by

$$ {}_a^RD_t^\alpha g(t) = D^m[D_t^{\alpha-m} g(t)] = D^m \frac{1}{\Gamma(m-\alpha)}\int_a^t (t-s)^{m-\alpha-1} g(t)ds, \qquad (2.2)$$

where, $D^m = \frac{d^m}{dt^m}$ is the standard m^{th} derivative (Saha, 2009).

2.3: Let $\alpha > 0$ and assume that g satisfies both ${}_a^RD_t^\alpha g$ and ${}_a^cD_t^\alpha g$, then

$$ {}_a^cD_t^\alpha g(t) = {}_a^RD_t^\alpha g(t) - \sum_{r=-0}^{m-1} \frac{g^{(r)}(a)}{\Gamma(r+1-\alpha)}(t-a)^{r-\alpha} \quad \text{(Mittal and Nigam, 2008)} \qquad (2.3)$$

2.4: Let $\alpha > 0,\ m-1<\alpha<m$, then

$$ {}_a^cD_t^\alpha g(t) = {}_a^RD_t^\alpha g(t) - \sum_{r=0}^{m-1} \frac{g^{(r)}(a)}{\Gamma(r+1-\alpha)}(t-a)^{r-\alpha}$$

$$= {}_a^RD_t^\alpha[g(t) - T_{n-1}[g,a](t)]. \text{ (Mittal and Nigam, 2008)}$$

2.5: Let $g(x) \in C_{-1}^{m}$ such that $m \in \mathbb{N} \cup \{0\}$. Then the Caputo fractional derivative of $g(x)$ satisfies the following properties for $m-1 < \alpha \leq m, m \in M,\ x > 0$. (Oyedepo *et al.*, 2016)

i. $D^{\alpha}C = 0,$ C is a constant.

ii. $D^{\alpha}x^{\beta} = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}x^{\beta-\alpha}, & \beta\epsilon\mathbb{N}, \quad \beta \leq \alpha_a \\ 0, & \beta\epsilon\mathbb{N}, \quad \beta < \alpha_a \end{cases},$

with $\alpha_a \geq \beta$ and $N = (0,1,2,3,\dots)$.

3. Orthogonal Polynomials

A sequence of polynomials $\{p_n(x)\}_0^{\infty}$ with degree $[p_n(x)] = n$ for each n is called orthogonal with respect to the weight function $w(x)$on interval $(a,b), \forall\ \alpha < b$ if

$$\int_a^b p_m(x)p_n(x)w(x) = \psi_g\delta_{mn} \text{ with } \delta_{mn} = \begin{Bmatrix} 0, m \neq n \\ 1, m = n \end{Bmatrix} \quad (3.1)$$

where δ_{mn} is called the Kronecker delta. The polynomial (3.1) is called orthogonal if $\psi_g = 0$ and orthonormal if $\psi_g = 1$. If $p_m(x) - k_m x^m +$ lower order terms with $K_m = 1$, for each $m \in \{0,1,2,3,\dots\}$ then the polynomials are called harmonic.

3.1. Construction of New Orthogonal Polynomials

We shall construct a new set of orthogonal polynomials with reference to the weight function, $w(x) = \cos \mathrm{x},\ x \in [0,1]$, via the following properties enlisted in Njoseh and Mamadu (2016) with little reformulation:

(i) $M_n(x) = \sum_{r=0}^{n} \mathrm{a_r}\, x^r$
(ii) $\langle M_m(x), M_n(x)\rangle = 0, m \neq n$ (3.1)
(iii) $M_n(1) = 1.$

The first seven constructed orthogonal polynomials $M_n(x)$, n $= 0,1,2,\dots$, are generated below via MAPLE 18 as

$M_0 \coloneqq 1$

$M_1 \coloneqq -0.8304877225 + 1.830487722 \cdot x$

$M_2 \coloneqq 0.7879437448 + -5.003808629 \cdot x + 5.215864884 \cdot x^2$

$M_3 \coloneqq 1.080849266 - 0.4399368024 \cdot x + 1.137426582 \cdot x^2 - 0.7783390452 \cdot x^3$

$M_4 \coloneqq 0.7630040138 - 15.74614028x + 72.85572149 \cdot x^2 - 115.9921591 \cdot x^3 + 59.11957391 \cdot x^4$

$$M_5 \coloneqq -0.7361927206 + 22.76946236 \cdot x - 163.7535856 \cdot x^2 + 446.9090052 \cdot x^3 - 512.7872919 \cdot x^4 + 208.5986027 \cdot x^5$$

$$M_6 \coloneqq -0.7958496274 + 25.31376421 \cdot x - 188.6937922 \cdot x^2 + 543.1910785 \cdot x^3 - 685.7621977 \cdot x^4 + 353.9749064 \cdot x^5 - 46.22790963 \cdot x^6$$

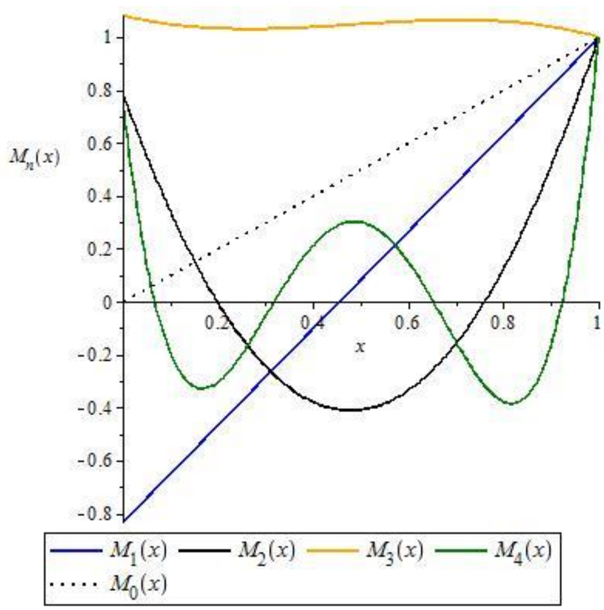

Figure 1. Graphical View of the $M_n(x), n = 0, 1, 2, \ldots$

3.2. Orthogonal Collocation Method (OCM)

Given the differential equation

$$N[y(x)] = 0, \tag{3.2}$$

where N is a differential operator.

Let the dependent variable $y(x)$ be expressed as

$$y(x) = \sum_{i=0}^{N} a_i y_i(x), \tag{3.3}$$

Where a_i, s are constants, and $y_i{}'s$ are polynomials. Thus, (3.2) becomes

$$N[\sum_{i=0}^{N} a_i y_i(x)] = 0 \tag{3.4}$$

called the residual equation. Now, the collocation method entails setting the residual to zero at the collocation points, that is,

$$N\left[\sum_{i=0}^{N} a_i y_i(x_j)\right] = 0, \quad j = 1, \ldots, N. \tag{3.5}$$

Equation (3.5) provides N equations in N unknowns.

The above procedure is more useful when the $y_i's$ are orthogonal polynomials, when collocation is performed at the zeros of $y_i's$, then the process is an orthogonal collocation method, as first used by Lanczos (1938).

3.3 Orthogonal Collocation Method for Fractional Fredholm Integro-Differential Equation

We present the orthogonal collocation method to study the approximate solution of the Fredholm fractional integro-differential equation as given below:

$$D^{\alpha}\big(u(x)\big) = g(x) + \int_a^b k(x,s)u(s)ds, \;\; x \geq -1, s \leq 1, \qquad (3.6)$$

with prescribed initial conditions

$$u^n(0) = B_n, \quad (n-1) < \alpha \leq n, \; n \in \mathbb{N}, \qquad (3.7)$$

where $D^{\alpha}\big(u(x)\big)$ is in Caputo sense, $g(x)$ is the source term, $u(x)$ is the unknown function, x and t are variables defined in the closed interval $[-1,1]$ and $k(x,t)$, being the kernel.

Equation (3.6) which has a unique solution under the following conditions:

i. $g(x)$ is a Sobolev space in $L_2([-1,1] \times [-1,1])$ and its norm is defined as

$$\| \; g(x) \; \| = \left[\int_a^b |f(x)|^2 dx\right]^{1/2} \leq \; B.$$

ii. kernel $k(x,s) \; \in \; L_2([-1,1] \times [-1,1]) \; \forall \, x,s \; \in \; [-1,1]$ and satisfies

$$|k(x,s)| \; \leq \; C,$$

where B and C are constants.

Using OCM in (3.6), we have

$$D^{\alpha}(G_N(x)) = g(x) + \int_a^b k(x,s)G_N(s)ds \; + \; E(x, a_1, a_2, \ldots, a_N)\,, \qquad (3.8)$$

where

$$G_N(x) = \sum_{r=0}^{N} a_r D^{\alpha}(M_r \; (x)), \qquad (3.9)$$

a_r are unknown constants and M_r are linearly independent functions called New Constructed polynomials. The error in (3.8) vanishes at N points $x_1\,, x_2\,, \ldots, x_N$ by collocating at the zeros of $M_r(x)$ for the unknown constants a_r (3.8) becomes

$$E(x, a_1, a_2, \ldots, a_N) = D^{\alpha}(G_N(x_r)) - g(x_r) - \int_a^b k(x_r, s) S_N(s) ds \ , \ 1 \leq r \leq N. \quad (3.10)$$

Thus, substituting the derived estimates a_r into (3.9), we obtain the approximate solution to (3.6)

3.4 Algorithm for Implementation of OCM

The implementation of the OCM is aided by the following steps:

i. Choose N arbitrarily in (3.9) and substitute it in (3.8).

ii. From item (i), introduce the Caputo fractional property on the variable x.

iii. Collocate orthogonally the resulting expansion from item (ii). The collocation points depend on the number of unknowns.

iv. Solve the system resulting from item (iii) via the Gaussian elimination method to obtain the values of $a_i, i = 0(1)n$.

v. Substitute the values of the $a_i, i = 0(1)n$, into (3.9) to obtain the approximate solution.

4. Numerical Illustrations

Example 4.1. (Mamadu *et al*., 2021)
Consider the following linear fractional integro-differential equation

$$D^{1/2}u(x) = \frac{(3/8)x^{3/2} - 2x^{1/2}}{\sqrt{\pi}} + \frac{x}{12} + \int_0^1 xtu(t)dt \, , x \geq 0, t \leq 1, \quad (4.1)$$

with initial condition$u(0) = 0$.

The analytic solution is given as $u(x) = x^2 - x$.

Let $N = 3$ in (3.9), we have that

$$u(x) = a_0 + a_1(-0.8304877225 + 1.830487722x) + a_2(0.7879437448 - 5.003808629x + 5.215864884x^2) + a_3(1.080849266 - 0.4399368024x + 1.137426582x^2 - 0.7783390452x^3) \quad (4.2)$$

Simplifying (4.3) in power of x, we have

$$u(x) = -0.7783390452a_3x^3 + (5.215864884a_2 + 1.137426582a_3)x^2 + (1.830487722a_1 - 5.003808629a_2 - 0.4399368024a_3)x + a_0 - 0.8304877225a_1 + 0.7879437448a_2 + 1.080849266a_3 \quad (4.3)$$

Substituting (4.3) into (4.1) and applying the Caputo property

$$D^{\alpha}x^{\gamma} = \begin{cases} 0, & \gamma \in \mathbb{N}_a, \gamma < \alpha_a \\ \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} x^{\gamma-a}, & \gamma \in \mathbb{N}_a, \gamma \geq \alpha_a \end{cases}$$

On the left-hand side, we get,

$$-\frac{2.490684945x^{5/2}\mathrm{a}_3}{\sqrt{\pi}}+\frac{8}{3}\frac{(5.215864884\mathrm{a}_2+1.137426582\mathrm{a}_3)x^{3/2}}{\sqrt{\pi}}$$

$$+\frac{2(1.830487722\mathrm{a}_1-5.003808629\mathrm{a}_2-0.4399368024\mathrm{a}_3)\sqrt{x}}{\sqrt{\pi}}-\frac{\frac{8}{3}x^{3/2}-2\sqrt{x}}{\sqrt{\pi}}-\frac{1}{12}x+0.1556678090\mathrm{a}_3x$$

$$-0.2500000000x-0.3333333333(1.830487722\mathrm{a}_1-5.003808629\mathrm{a}_2-0.4399368034\mathrm{a}_3)$$

$$-0.5000000000x(\mathrm{a}_0-0.8304877225\mathrm{a}_1+0.7879437448\mathrm{a}_2+1.080849266\mathrm{a}_3)=0 \tag{4.4}$$

Since there are four unknowns, we collocate orthogonally at the zeros of $M_4(x)$, that is, 0.06718313659, 0.3187509266, 0.6526993486, and 0.9233590625. Thus, we obtain the following systems of linear equations:

Collocating at $t=0.067183$, we have:

$$-0.03359156830\mathrm{a}_0+0.5222722669\mathrm{a}_1-1.328842294a_2-0.1356150621a_3+0.2606750435=0$$

Collocating at $t=0.318751$, we have:

$$-0.1593754633a_0+1.104001196a_1-1.785089162a_2-0.2194504687a_3+0.3397465151=0$$

Collocating at $t=0.652699$, we have:

$$-0.3263496743a_0+1.541477464a_1-0.443136783a_2-0.3233379572a_3+0.06387608292=0$$

Collocating at $t=0.923359$, we have:

$$-0.4616795312a_0+1.804776332a_1+1.509455215a_2-0.5923367408a_3-0.3275723178=0$$

$a_0 = -0.169512, a_1 = -0.022210, a_2 = 0.191723$, and $a_3 = 0.000000$.

Substituting into (4.2), we have our approximate solution as

$$u(x)=7.09\ 10^{-8}-1.000000334x+1.000001263x^2 \tag{4.5}$$

Example 4.2. (Mamadu *et al*., 2021)

Consider the following linear fractional integro-differential equation

$$D^{5/3}u(x)=\frac{3\sqrt{3}\Gamma(2/3)}{\pi}-\frac{1}{5}x^2-\frac{1}{4}x+\int_0^1(xt+x^2t^2)u(t)dt\,, x\geq 0, t\leq 1, \tag{4.6}$$

with initial conditions $u(0)=u'(0)=0$.

The analytic solution is $u(x)=x^2$.

Solving (4.6) in similar fashion as Example 4.1, we have our approximate solution as

$$u(x)=-3.29\ 10^{-8}-0.0000012306x+1.000001263x^2 \tag{4.7}$$

5. Tables and Graphical Representation of Results

We now present the tables of exact and approximate results with figures to enable us compare our results with those in literature.

Table 1: Comparison of Results between the exact solution, New Method approximate Solution and Mamadu *et al.* (2021) via Galerkin method for Example 4.1

x	Exact	Approximate of the New method	New Method Error	Mamadu *et al.* (2021) Error
0.00	0.0000000	0.0000000	3.8159e-09	0.0000000
0.10	-0.0900000	-0.0900000	3.8200e-09	0.0000000
0.20	-0.1600000	-0.1600000	3.8000e-09	0.0000000
0.30	-0.2100000	-0.2100000	4.0000e-09	0.0000000
0.40	-0.2400000	-0.2400000	4.2000e-09	0.0000000
0.50	-0.2500000	-0.2500000	4.4000e-09	0.0000000
0.60	-0.2400000	-0.2400000	4.6000e-09	0.0000000
0.70	-0.2100000	-0.2100000	5.0000e-09	0.0000000
0.80	-0.1600000	-0.1600000	5.2000e-09	0.0000000
0.90	-0.0900000	-0.0900000	5.4200e-09	0.0000000
1.00	0.0000000	0.0000000	5.8159e-09	0.0000000

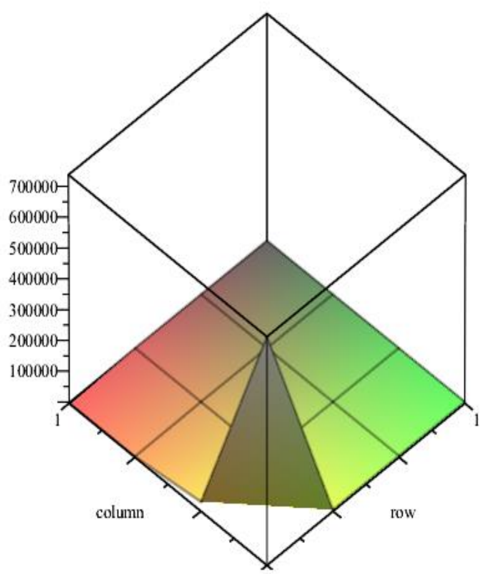

Figure 2a. Matrix inverse of Example 4.1.

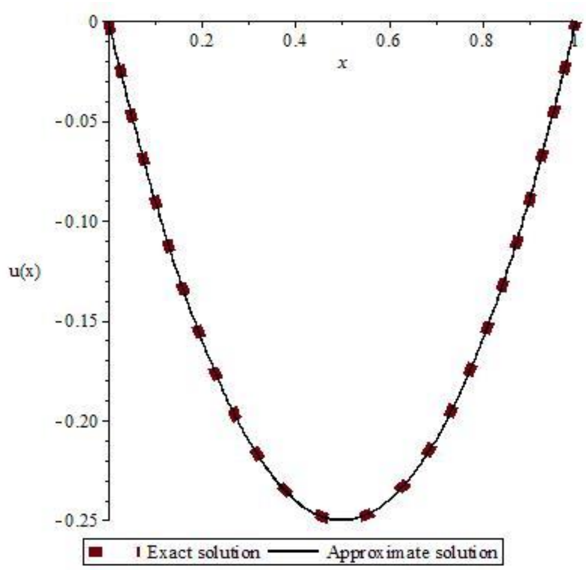

Figure 2b. Approximate solution of Example 1 as compared with exact solution.

Table 2: Comparison of Results between the Exact Solution, New method approximate solution and Mamadu *et al.* (2021) via Galerkin method for Example 4.2

x	Exact	Approximate of the proposed method	Error from New Method	Mamadu *et al.* (2021) Error
0.00	0.0000000	-0.0000001	7.8764e-08	0.0000000
0.10	0.0100000	0.0099999	6.5464e-08	0.0000000
0.20	0.0400000	0.0399999	5.2160e-08	0.0000000
0.30	0.0900000	0.0900000	3.8860e-08	0.0000000
0.40	0.1600000	0.1600000	2.5500e-08	0.0000000
0.50	0.2500000	0.2500000	1.2100e-08	0.0000000
0.60	0.3600000	0.3600000	1.4000e-09	0.0000000
0.70	0.4900000	0.4900000	1.4900e-08	0.0000000
0.80	0.6400000	0.6400000	2.8600e-08	0.0000000
0.90	0.8100000	0.8100000	4.2800e-08	0.0000000
1.00	1.0000000	1.0000001	5.6000e-08	0.0000000

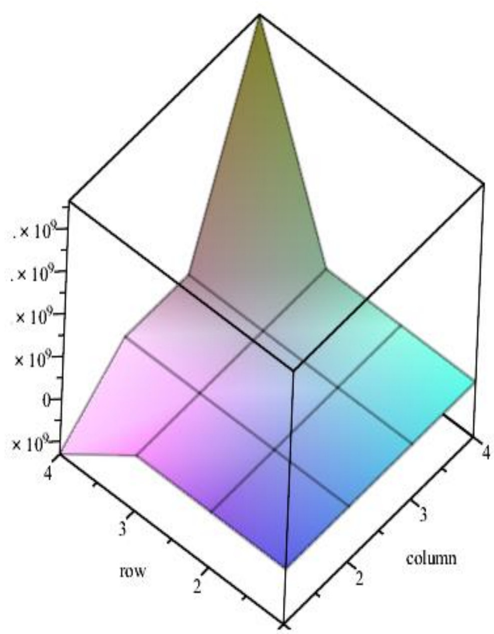

Figure 3a. Matrix inverse of Example 4.2.

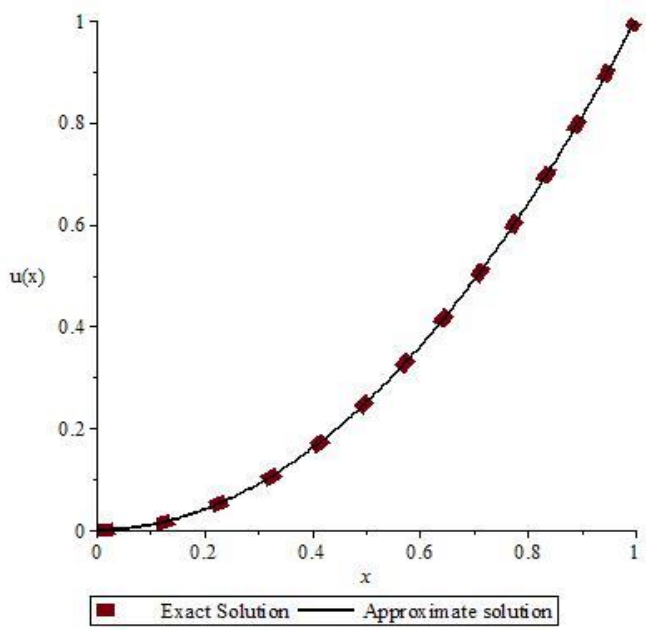

Figure 3b. Approximate solution of Example 2 as compared with exact solution

6. Discussion of Results

The use of orthogonal polynomials as basis functions via a suitable approximation scheme for the solution of many problems in science and technology has been on the increase and quite fascinating. In many numerical schemes, the convergence depends solely on the nature of the basis function adopted. The New Constructed polynomials are orthogonal polynomials developed in this present study with reference to the weight function, $w(x) = \cos x, x \in [0,1]$. We have successfully implemented the OCM for the solution of fractional Fredholm integro-differential equation using New Constructed polynomials as basis functions. The resulting numerical evidence shows that the method derives accurate and reliable approximation with an excellent convergent rate for both illustrations considered with results presented in graphs and Tables and are also compared with those available in the literature. Specifically, the constructed polynomials exhibit rapid convergence for both examples considered, attaining maximum errors of order 10^{-9} and 10^{-8}, as shown in Tables 1 and 2, respectively. When compared with the Mamadu-Njoseh polynomials, it is obvious the newly constructed polynomials are less impressive in terms of convergence. The reason is not far-fetched. Mamadu-Njoseh polynomials are defined in terms of an algebraic weight function, while the newly constructed polynomials are defined in trigonometric weight function. Also, the graphical illustration in Figures 2a, 2b, 3a, and 3b, shows the degree of convergence between the exact solution and the computed solution. All computational frameworks are performed via MAPLE.

7. Conclusion

This research has implemented an accurate, valid, and reliable numerical procedure for the solution of the fractional Fredholm integro-differential equation. We have also considered an approximate formulation in the Caputo sense in terms of the New Constructed polynomials. The results reveal that the procedure converges faster even as N increases. We expressed the solution as a truncated orthogonal series so that it can easily be solved by any mathematical software without any computational stress. From the examples considered above, it is evident that the method is very accurate as it converges rapidly to the analytic solution. We have also presented our numerical evidence graphically showing the comparison of solutions between the analytic and approximate.

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