TAU-COLLOCATION APPROACH FOR SOLVING FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract

This work is motivated by the desire to cover the gap in the literature by seeking the numerical solutions to fractional integro-differential equations using the class of orthogonal polynomials called "the Mamadu-Njoseh polynomials" as trial solutions, using the Tau-Collocation method. We considered the Tau-Collocation method for solving the fractional Integro-differential equation (FIDE) (Caputo-sense) with aid of Mamadu-Njoseh and Chebyshev polynomials. Numerical examples solved via MAPLE 18 showed that the Tau-Collocation method is an excellent solver of fractional integro-differential equations. Also, the absolute convergence of the method shows the effectiveness of Mamadu-Njoseh and Chebyshev orthogonal polynomials as basis functions for the fractional integro-differential equations.

Keywords: Fractional derivative, Tau Method, Collocation Method, Orthogonal Polynomials.

1. Introduction

The role of fractional derivatives in modern science and technology is quite fascinating. It has found applications in Biology, Physics, Economics, and fluid mechanics. Many physical phenomena are governed by fractional differential equations (FDEs), which have attracted the attention of many researchers. Quite a number of FDEs do not possess analytic solutions, giving room to numerical procedures. Some of the available numerical techniques for solving FDEs include the Homotopy perturbation method, Adomian decomposition method, Variational iteration method, Collocation method, and Homotopy analysis method, among others. Several forms of fractional integro-differential equations had been proposed in standard models, and there had been significant interest in developing numerical schemes for their solution (Edwards *et al.*, 2002).

The Tau collocation method was introduced by Lanczos, to provide approximate polynomial solutions for linear ordinary differential equations with polynomial coefficients. The method takes advantage of the special properties of the Chebychev polynomial (Oritz, 1969; Oritz, 1975; Oritz, 1979; Coleman, 1974 and Khajah, 1999). Tau-collocation was initially formulated as a tool for the approximation of special functions of mathematical physics, which could be expressed in terms of simple differential equations. The interest in the Tau method for a long period of time is regarded only as a tool for the construction of accurate approximations of a very restricted class of functions. This has been enhanced by the availability of the software for its computer implementation and by the possibility of using it in the numerical solution of complex nonlinear differential equation intervals. The approximation of the solution of such type of equations is achieved as a result of finding Tau approximants of a sequence of problems defined by linear differential equations (Oritz *et al.,* 1978; Oritz and Samara, 1981; Oritz and Samara, 1984; Mamadu and Njoseh, 2016a).

The Integro-differential equation (IDE) is one that considers both integrals and derivatives of an unknown function. In other words, the integro-differential equation is an equation where both differential and integral operators will appear in the same equation. Integro-differential equations are usually difficult to solve analytically; hence solution methods in the literature involve search for efficient approximate solutions (Babolian *et al.*, 2007). Recently, several numerical methods to solve IDEs have been proposed; these include the Wavelet-Galerkin Method (Avudainayagam, and Vani, 2000), Variational Iteration Method (Njoseh and Mamadu, 2016a), Orthogonal Collocation Methods (Mamadu and Njoseh, 2016b), Variation Iteration Decomposition Method (Njoseh and Mamadu, 2016b), Modified Variational Homotopy Perturbation Method (Njoseh and Mamadu, 2016c), Homotopy Perturbation Method (Sweilam *et al.*, 2008; Saeed and Sdeq, 2010; Oyedepo, 2019), and Tau Method (Khajah, 1999; Hosseini and Shahmorad, 2003a; Hosseini and Shahmorad, 2003b), among others.

In this study, we intend to apply the Mamadu-Njoseh and Chebyshev polynomials as basis functions in resolving the approximate solution of FIDEs using the Tau Collocation method and compare the results obtained from both sets of polynomials to determine which converges faster to the exact solution.

2. Preliminaries

Definition 2.1. (Caputo Fractional Derivative)

The first or left-sided Caputo derivative is given as (Mamadu *et. al*., 2021)

$$ {}_aD_x^{\alpha} g(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} g^n(t)\,dt, & (n-1) < \alpha \le n \\ \left(\frac{d}{dx}\right)^{n-1} g(t), & if \quad \alpha + 1 = n \end{cases} \tag{1} $$

where $g^n(t)$ denote the nth integer derivative of $g(t)$.

Useful Properties of the Caputo Fractional Derivative

Some properties of Caputo fractional derivative used in this work are stated as follows (Mamadu *et. al*., 2021)

i. $D^{\alpha} K = 0$, K is a constant

ii. $D^{\alpha} x^m = \begin{cases} 0, & m \in N, m \ge (\alpha) \\ \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} x^{\beta-\alpha}, & if \;\; m \in N \; and \;\; m < (\alpha), \end{cases}$ where $(\alpha) \ge \alpha$ and $N = \{0, 1, 2, \dots\}$

iii. $D^{\alpha}\big(\lambda g_1(x) + \mu g_2(x)\big) = \lambda D^{\alpha} g_1(x) + \mu D^{\alpha} g_2(x)$

Definition 2.2. (Fractional Integro-Differential Equation)

A fractional integro-differential equation has the form (Mohammed *et al*., 2020)

$$\begin{cases} D^{\alpha}u(x) = g(x) + \int_0^1 k(x,s)u(s)ds, & x \geq 0, s \leq 1, \\ u^i(0) = \beta_i, \quad (n-1,n] \in \alpha, \ n \in \mathbb{N}, \ i = 1, \cdots, n \end{cases} \tag{4}$$

where $u(x)$ is the unknown, $D^{\alpha}u(x)$ is the Caputo fractional derivative of $u(x)$ of order α, $g(x)$ is the non-homogeneous term, $k(x,s)$ is the nucleus of the integral, x and s are variables defined in [0,1].

3. Tau-Collocation Method

A fractional Fredholm integro-differential equation has the form (Mohammed *et al*., 2020)

$$D^{\alpha}u(x) = r(x) + \int_a^b k(x,t)u(t)dt\,, \quad x \geq a, t \leq b, \tag{5}$$

with conditions

$$u^i(0) = \beta_i, \quad (n-1,n) \in \alpha, \ n \in N, i = 1, \cdots, n$$

where $D^{\alpha}u(x)$ denotes the αth Caputo fractional derivative of $u(x)$, $r(x)$ is the source term, $k(x,t)$ is the kernel, x and t are variables defined in $[a,b]$, and $u(x)$ is the required function to be estimated.

Let us assume an approximation of the form

$$L[u_n(x)] = \sum_{r=0}^{m} a_r \gamma_r(x), \tag{6}$$

where $a_r{}'s$ are unknown constants to be estimated, $\gamma_r{}'s$ are orthogonal polynomials, and L is a linear differential operator.

Consider an approximation to the residual, R_n as

$$R_n = \tau_1 T_{n-2}(x) + \tau_2 T_{n-1}(x) + \cdots + \tau_m T_{n-m+1}(x)\,.$$

Then by the Tau-Collocation method if

$L[y(x)] = f(x)$,

$$\Rightarrow L[y(x)] = f(x) + R_n = f(x) + \tau_1 T_{n-2}(x) + \tau_2 T_{n-1}(x) + \cdots + \tau_m T_{n-m+1}(x). \tag{7}$$

Thus, (7) becomes

$$D^{\alpha}(\sum_{r=0}^{n} a_r \gamma_r(x),) = r(x) + \int_a^b k(x,t)(\sum_{r=0}^{n} a_r \gamma_r(t))dt\,, \quad x \geq a, t \leq b, \tag{8}$$

Now, collocating (8) at equidistant points, that is, $x_k = \left(\frac{b-a}{N+1}\right)k, \ k > 0$, to get

$$D^{\alpha}\left(\sum_{r=0}^{n} a_r \gamma_r(x_k)\,,\right) - r(x_k) - \int_a^b k(x_k,t)\left(\sum_{r=0}^{n} a_r \gamma_r(t_k)\,,\right)dt = \tau_1 T_{n-2}(x)$$

$$+\tau_2 T_{n-1}(x) + \cdots + \tau_m T_{n-m+1}(x) \tag{9}$$

The resulting $(n+1)$ linear equations from (9) are thus solved by Gaussian elimination to arrive at the required approximate solution.

4. Numerical Examples

Example 4.1 Consider the following linear fractional integro-differential equation (Njoseh and Mamadu, 2016c)

$$D^{\frac{1}{2}}u(x) = \frac{\left(\frac{3}{8}\right)x^{\frac{3}{2}} - 2x^{\frac{1}{2}}}{\sqrt{\pi}} + \frac{x}{12} + \int_0^1 xtu(t)dt\,, x \geq 0, t \leq 1, \tag{10}$$

with initial $u(0) = 0$. The analytic solution is given as $u(x) = x(x-1)$.

Case 1: Using Mamadu-Njoseh Basis Functions

Let $n = 3$ in (7), we have that

$$u(x) = \frac{7}{20}a_3x^3 + \left(\frac{5}{12}a_2 + \frac{21}{20}a_3 + \frac{5}{12}\tau_1\right)x^2 + \left(\frac{1}{2}a_1 + \frac{5}{6}a_2 + \frac{3}{20}a_3 + \frac{5}{6}\tau_1\right)x$$
$$+a_0 + \frac{1}{2}a_1 - \frac{1}{4}a_2 - \frac{11}{20}a_3 - \frac{1}{4}\tau_1 \tag{11}$$

Substituting (11) into (10) and applying the Caputo property

$$D^{\alpha}x^{\gamma} = \begin{cases} 0, & \gamma \in N_a, \\ \gamma < \alpha_a \ \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma - \square + 1)}x^{\gamma - a}, & \gamma \in \square_a, \gamma \geq \alpha_a \end{cases}$$

on the left-hand side, we get,

$$\frac{28}{25}\frac{a_3x^{5/2}}{\sqrt{\pi}} + \frac{8}{3}\frac{\left(\frac{5}{12}a_2 + \frac{21}{20}a_3 + \frac{5}{12}\tau_1\right)x^{3/2}}{\sqrt{\pi}}$$
$$+\frac{2\left(\frac{1}{2}a_1 + \frac{5}{6}a_2 + \frac{3}{20}a_3 + \frac{5}{6}\tau_1\right)\sqrt{x}}{\sqrt{\pi}} - \frac{\frac{8}{3}x^{\frac{3}{2}} - 2\sqrt{x}}{\sqrt{\pi}} - \frac{1}{2}x$$
$$-\frac{1}{2}x\left(\frac{7}{20}a_3x^3 + \left(\frac{5}{12}a_2 + \frac{21}{20}a_3 + \frac{5}{12}\tau_1\right)x^2 + \left(\frac{1}{2}a_1 + \frac{5}{6}a_2 + \frac{3}{20}a_3 + \frac{5}{6}\tau_1\right)x\right)$$
$$+a_0 + \frac{1}{2}a_1 - \frac{1}{4}a_2 - \frac{11}{20}a_3 - \frac{1}{4}\tau_1 = 0 \tag{12}$$

Now, collocate (12) at the equidistant points, that is, $x_k = \left(\frac{b-a}{n+1}\right)k$ with $n = 3,\ k = 1,2,3,4$, to obtain the following systems of equations:

$$A_1 \coloneqq \frac{1}{38918880000}\frac{1}{\pi^{\frac{3}{2}}}(64868000a_0\pi^{\frac{3}{2}} + 1081080000\pi^{\frac{3}{2}} - 45491846400a_3\pi^{\frac{3}{2}}$$
$$-38301120000a_2\pi^{\frac{3}{2}} - 1081080\pi\tau_1 - 15567552000a_1\pi - 148262400$$
$$+5675670000a_1\pi^{\frac{3}{2}} + 405050000a_2\pi^{\frac{3}{2}} + 2383781400a_3\pi^{\frac{3}{2}}$$
$$+51068160\tau_1\pi^{\frac{3}{2}}) = 0$$

$$A_2 \coloneqq \frac{1}{38918880000}\frac{1}{\pi^{\frac{3}{2}}}(5027022000a_1\pi^{\frac{3}{2}} + 945945000\pi^{\frac{3}{2}} - 42199872000\pi a_2$$
$$-42199872000\pi\tau_1 - 15567552000a_0\pi + 38918880000\sqrt{\pi}a_1 - 26687232000a_1\pi$$
$$-2924064000\pi + 17297280\pi a_0 - 149909760\sqrt{\pi}a_3$$
$$-124924800\sqrt{\pi}a_2 + 93693600000\sqrt{\pi}a_2 + 106118812800\sqrt{\pi}a_3$$
$$+93693600000\sqrt{\pi}\tau_1 - 45148147200\pi a_3 + 5675670000a_0\pi^{\frac{3}{2}}$$
$$+3716212500a_2\pi^{\frac{3}{2}} + 2335132800a_3\pi^{\frac{3}{2}} + 3716212500\tau_1\pi^{\frac{3}{2}} + 8648640000\sqrt{\pi}) = 0$$

$$A_3 \coloneqq \frac{1}{38918880000}\frac{1}{\pi^{\frac{3}{2}}}(3005145000a_2\pi^{\frac{3}{2}} + 675675000\pi^{\frac{3}{2}} - 42199872000a_1\pi$$
$$-38301120000a_0\pi + 228228000000\sqrt{\pi}a_2 - 29343600\pi a_2 - 4671264000\pi$$
$$+93693600000\sqrt{\pi}a_1 + 265426761600\sqrt{\pi}a_3 + 228228000000\sqrt{\pi}\tau_1$$
$$-43324934400\pi a_3 - 46317440000\pi\tau_1 + 4054050000a_0\pi^{\frac{3}{2}} + 3716212500a_2\pi^{\frac{3}{2}}$$
$$+2191214025a_3\pi^{\frac{3}{2}} + 3005145000\tau_1\pi^{\frac{3}{2}} \mp 14414400000\sqrt{\pi}) = 0$$

$$A_4 \coloneqq \frac{1}{38918880000}\frac{1}{\pi^{\frac{3}{2}}}(1934322390a_3\pi^{\frac{3}{2}} + 397296900\pi^{\frac{3}{2}} - 45491846400a_0\pi$$
$$+326643565248\sqrt{\pi}a_3 - 4356835200\pi + 106118812800\sqrt{\pi}a_1 + 265426761600\sqrt{\pi}a_2$$
$$+265426761600\sqrt{\pi}\tau_1 - 43324934400\pi a_2 - 43324934400\pi\tau_1 - 45148147200\pi a_1$$
$$+2383781400a_0\pi^{\frac{3}{2}} + 2335132800a_1\pi^{\frac{3}{2}} + 2191214025a_2\pi^{\frac{3}{2}} + 2191214025\tau_1\pi^{\frac{3}{2}}$$
$$-39031151616\pi a_3 - 311351040\sqrt{\pi}) = 0$$

Using the initial condition $u(0) = 0$, in (12), we obtain,

$$A_5 \coloneqq a_0 + \frac{1}{2}a_1 - \frac{1}{4}a_2 - \frac{11}{20}a_3 - \frac{1}{4}\tau_1 = 0$$

Solving the above equations via MAPLE 18, we obtain values

$$a_0 = 3.600000, a_1 = -6.000000, a_2 = 2.400000, a_3 = 0.000000, \tau = 0.000000.$$

Substituting into (7), we have the approximate solution to (10) as

$$u(x) = x^2 - x,$$

which is the exact solution itself.

The computational results are shown in Table 1.

Case 2: Using Chebyshev Basis Functions of the first kind

Let $n = 3$ in (7), we have that

$$u(x) = \frac{1}{2}a_3x^3 + \left(\frac{1}{2}a_2 + \frac{3}{2}a_3 + \frac{1}{2}\tau_1\right)x^2 + \left(\frac{1}{2}a_1 + a_2 + \tau_1\right)x$$
$$+a_0 + \tfrac{1}{2}a_1 - \tfrac{1}{2}a_2 - a_3 - \tfrac{1}{2}\tau_1 \qquad (13)$$

Substituting (13) into (10) and applying the Caputo property

$$D^\alpha x^\gamma = \begin{cases} 0, & \gamma \in N_a, \\ \gamma < \alpha_a \ \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma - \square + 1)} x^{\gamma - a}, & \gamma \in \square_a, \gamma \geq \alpha_a \end{cases}$$

on the left-hand side, we get,

$$\frac{8}{5}\frac{a_3x^{5/2}}{\sqrt{\pi}} + \frac{8}{3}\frac{\left(\frac{1}{2}a_2 + \frac{3}{2}a_3 + \frac{1}{2}\tau_1\right)x^2}{\sqrt{\pi}}$$
$$+\frac{2\left(\frac{1}{2}a_1 + a_2 + \tau_1\right)\sqrt{x}}{\sqrt{\pi}} - \frac{\frac{8}{3}x^{\frac{3}{2}} - 2\sqrt{x}}{\sqrt{\pi}} - \frac{1}{12}x$$
$$-\frac{1}{2}x\left(\frac{1}{2}a_3x^3 + \left(\frac{1}{2}a_2 + \frac{3}{2}a_3 + \frac{1}{2}\tau_1\right)x^2 + \left(\frac{1}{2}a_1 + a_2 + \tau_1\right)x\right)$$
$$+a_0 + \tfrac{1}{2}a_1 - \tfrac{1}{2}a_2 - a_3 - \tfrac{1}{2}\tau_1 = 0 \qquad (29)$$

Now, collocate (13) at the equidistant points, that is, $x_k = \left(\frac{b-a}{n+1}\right)k$ with $n = 3,\ k = 1,2,3,4$, to obtain the following systems of equations:

$$A_1 := \frac{1}{43243200}\frac{1}{\pi^{\frac{3}{2}}}(-7207200a_0\pi^{\frac{3}{2}} - 1201200\pi^{\frac{3}{2}} - 6306300a_1\pi^{\frac{3}{2}}$$
$$-3963960a_2\pi^{\frac{3}{2}} - 1081080a_3\pi^{\frac{3}{2}} - 3963960\tau_1\pi^{\frac{3}{2}} + 1647360\pi + 51068160a_2\pi$$
$$+51068160\pi\tau_1 + 17297280\pi a_1 + 64796160\pi a_3) = 0$$

$$A_2 := \frac{1}{43243200}\frac{1}{\pi^{\frac{3}{2}}}(-5585580a_1\pi^{\frac{3}{2}} - 1051050\pi^{\frac{3}{2}} - 6306300a_0\pi^{\frac{3}{2}}$$

$$-3963960a_2\pi^{\frac{3}{2}} - 1312740a_3\pi^{\frac{3}{2}} - 3963960\tau_1\pi^{\frac{3}{2}} - 43243200\sqrt{\pi}a_1 + 296524800a_1\pi$$
$$+3248960\pi + 17297280\pi a_0 - 149909760\sqrt{\pi}a_3 - 124924800\sqrt{\pi}a_2 - 124924800\sqrt{\pi}\tau_1$$
$$+58955520\pi a_3 + 52807040\pi a_2 - 52807040\pi\tau_1 - 9609600\sqrt{\pi}) = 0$$

$$A_3 := \frac{1}{43243200}\frac{1}{\pi^{\frac{3}{2}}}(-2934360a_2\pi^{\frac{3}{2}} - 660660\pi^{\frac{3}{2}} - 3963960a_0\pi^{\frac{3}{2}}$$
$$-3963690a_1\pi^{\frac{3}{2}} - 1833975a_3\pi^{\frac{3}{2}} - 29343600\tau_1\pi^{\frac{3}{2}} - 365164800\sqrt{\pi}a_2 + 53680640a_1\pi$$
$$+58988800\pi + 51068160\pi a_0 - 124924800\sqrt{\pi}a_1 - 452035584\sqrt{\pi}a_3 - 365164800\sqrt{\pi}\tau_1$$
$$+45450240\pi a_3 + 52807040\pi a_1 + 53680640\pi\tau_1 - 19219200\sqrt{\pi}) = 0$$

$$A_4 := \frac{1}{43243200}\frac{1}{\pi^{\frac{3}{2}}}(-2235090a_3\pi^{\frac{3}{2}} - 180180\pi^{\frac{3}{2}} - 1081080a_0\pi^{\frac{3}{2}}$$
$$-1312740a_1\pi^{\frac{3}{2}} - 1833975a_2\pi^{\frac{3}{2}} - 1833975\tau_1\pi^{\frac{3}{2}} - 604251648\sqrt{\pi}a_3 + 5523200\pi$$
$$-149909760\sqrt{\pi}a_1 - 452035584\sqrt{\pi}a_2 - 452035584\sqrt{\pi}\tau_1 + 45450240\pi a_3$$
$$+45450240\pi\tau_1 + 58955520\pi a_1 + 6479610\pi a_0 + 32526336\pi a_3 + 4612608\sqrt{\pi}) = 0$$

Using the initial condition $u(0) = 0$, in (13), we obtain,

$$A_5 := a_0 + \frac{1}{2}a_1 - \frac{1}{2}a_2 - a_3 - \frac{1}{2}\tau_1 = 0$$

Solving the above equations via MAPLE 18 , we obtain values

$a_0 = 3.600000, a_1 = -6.000000, a_2 = 2.400000, a_3 = 0.000000, \tau = 0.000000.$

Substituting into (22), we have the approximate solution to (25) as

$$u(x) = x^2 - x,$$

which is the exact solution itself.

The computational results are shown in Table 1.

Example 4.2: Consider the following linear fractional integro-differential equation (Njoseh and Mamadu, 2016c)

$$D^{\frac{5}{3}}u(x) = \frac{3\sqrt{3}\Gamma\left(\frac{2}{3}\right)}{\pi} - \frac{1}{5}x^2 - \frac{1}{4}x + \int_0^1 (xt + x^2t^2)u(t)dt\,, x \geq 0, t \leq 1, \qquad (15)$$

subject to $u(0) = u'(0) = 0$. The exact solution is given as $u(x) = x^2$.

Case 1: Using Mamadu-Njoseh Basis Functions

Let $n = 2$ in (7), we have that

$$u(x) = \left(\frac{5}{12}a_2 + \frac{5}{12}\tau_1 + \frac{5}{12}\tau_2\right)x^2 + \left(\frac{1}{2}a_1 + \frac{5}{6}a_2 + \frac{5}{6}\tau_1 + \frac{5}{6}\tau_2\right)x$$
$$+a_0 + \tfrac{1}{2}a_1 - \tfrac{1}{4}a_2 - \tfrac{1}{4}\tau_1 - \tfrac{1}{4}\tau_2 \qquad (16)$$

Substituting (16) into (15) and applying the Caputo property

$$D^{\alpha}x^{\gamma} = \begin{cases} 0, & \gamma \in N_a, \\ \gamma < \alpha_a \ \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma - \square + 1)}x^{\gamma - a}, & \gamma \in \square_a, \gamma \geq \alpha_a \end{cases}$$

on the left-hand side, we get,

$$\frac{8}{3}\frac{\left(\frac{5}{12}a_2 + \frac{5}{12}\tau_1 + \frac{5}{12}\tau_2\right)x^{3/2}}{\sqrt{\pi}}$$
$$+\frac{2\left(\frac{1}{2}a_1 + \frac{5}{6}a_2 + \frac{5}{6}\tau_1 + \frac{5}{6}\tau_2\right)\sqrt{x}}{\sqrt{\pi}} - \frac{\frac{8}{3}x^{\frac{3}{2}} - 2\sqrt{x}}{\sqrt{\pi}} - \frac{1}{12}x$$
$$-\frac{1}{2}x\left(\left(\frac{5}{12}a_2 + \frac{5}{12}\tau_1 + \frac{5}{12}\tau_2\right)x^2 + \left(\frac{1}{2}a_1 + \frac{5}{6}a_2 + \frac{5}{6}\tau_1 + \frac{5}{6}\tau_2\right)x\right)$$
$$+a_0 + \tfrac{1}{2}a_1 - \tfrac{1}{4}a_2 - \tfrac{1}{4}\tau_1 - \tfrac{1}{4}\tau_2 = 0 \qquad (17)$$

Now, collocate (17) at the equidistant points, that is, $x_k = \left(\frac{b-a}{n+1}\right)k$ with $n = 2,\ k = 1, 2, 3$, to obtain the following systems of equations:

$$A_1 := \frac{1}{5987520}\frac{1}{\pi^{\frac{3}{2}}}(997920a_0\pi^{\frac{3}{2}} + 166320\pi^{\frac{3}{2}} + 873180a_1\pi^{\frac{3}{2}} + 623700a_2\pi^{\frac{3}{2}} + 623700\tau_1\pi^{\frac{3}{2}}$$
$$+623700\tau_2\pi^{\frac{3}{2}} - 2395008\pi a_1 - 5892480a_2\pi + 51068160\pi\tau_1 - 5892480\pi\tau_2) = 0$$

$$A_2 := \frac{1}{5987520}\frac{1}{\pi^{\frac{3}{2}}}(7733880a_1\pi^{\frac{3}{2}} + 145530\pi^{\frac{3}{2}} + 873180a_0\pi^{\frac{3}{2}} + 571725a_2\pi^{\frac{3}{2}}$$
$$+571725\tau_1\pi^{\frac{3}{2}} + 571725\tau_2\pi^{\frac{3}{2}} + 2395008a_0\pi - 4105728a_1\pi + 4498560\pi$$
$$+5987520\sqrt{\pi}a_1 + 14414400\sqrt{\pi}a_2 + 14414400\sqrt{\pi}\tau_1 + 14414400\sqrt{\pi}\tau_2$$
$$-6492288\pi a_2 - 6492288\pi\tau_1 - 6492288\pi\tau_2 + 1330560\sqrt{\pi}) = 0$$

$$A_3 := \frac{1}{5987520}\frac{1}{\pi^{\frac{3}{2}}}(-2235090a_3\pi^{\frac{3}{2}} + 103950\pi^{\frac{3}{2}} + 623700a_0\pi^{\frac{3}{2}} + 5671725\pi^{\frac{3}{2}}$$

$$+462330\tau_1\pi^{\frac{3}{2}} - 7125760a_2\pi - 718656\pi + 14414400\sqrt{\pi}a_1 + 35112000\sqrt{\pi}\tau_2$$
$$-7125760\pi\tau_1 - 7125760\pi\tau_2 + 45450240\pi a_1 - 58955520\pi a_0 + 2217600\sqrt{\pi}) = 0$$

Using the initial condition on $u(0) = u'(0) = 0$, on (17), we obtain,

$$A_4 \coloneqq a_0 + \frac{1}{2}a_1 - \frac{1}{4}a_2 - \frac{1}{4}\tau_1 - \frac{1}{4}\tau_2 = 0$$
$$A_5 \coloneqq \frac{1}{2}a_1 + \frac{5}{6}a_2 + \frac{5}{6}\tau_1 + \frac{5}{6}\tau_2 = 0$$

Solving the above equations via MAPLE 18, we obtain values

$a_0 = 2.600000, a_1 = -4.000000, a_2 = 2.400000, a_3 = 0.000000, \tau_1 = 0.000000,$
$\tau_2 = -0.000000.$

Substituting into (17), we have the approximate solution to (10) as

$$u(x) = x^2.$$

The computational results are shown in Table 2.

Case 2: Using Chebyshev Basis Functions of the first kind

Let $n = 2$ in (7), we have that

$$u(x) = \left(\frac{5}{12}a_2 + \frac{5}{12}\tau_1 + \frac{5}{12}\tau_2\right)x^2 + \left(\frac{1}{2}a_1 + \frac{5}{6}a_2 + \frac{5}{6}\tau_1 + \frac{5}{6}\tau_2\right)x$$
$$+a_0 + \tfrac{1}{2}a_1 - \tfrac{1}{4}a_2 - \tfrac{1}{4}\tau_1 - \tfrac{1}{4}\tau_2 \tag{16}$$

Substituting (16) into (15) and applying the Caputo property

$$D^\alpha x^\gamma = \begin{cases} 0, & \gamma \in N_a, \\ \gamma < \alpha_a \ \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma - \square + 1)} x^{\gamma - a}, & \gamma \in \square_a, \gamma \geq \alpha_a \end{cases}$$

on the left-hand side, we get,

$$\frac{8}{3}\frac{\left(\frac{5}{12}a_2 + \frac{5}{12}\tau_1 + \frac{5}{12}\tau_2\right)x^{3/2}}{\sqrt{\pi}}$$
$$+\frac{2\left(\frac{1}{2}a_1 + \frac{5}{6}a_2 + \frac{5}{6}\tau_1 + \frac{5}{6}\tau_2\right)\sqrt{x}}{\sqrt{\pi}} - \frac{\frac{8}{3}x^{\frac{3}{2}} - 2\sqrt{x}}{\sqrt{\pi}} - \frac{1}{12}x$$
$$-\frac{1}{2}x\left(\left(\frac{5}{12}a_2 + \frac{5}{12}\tau_1 + \frac{5}{12}\tau_2\right)x^2 + \left(\frac{1}{2}a_1 + \frac{5}{6}a_2 + \frac{5}{6}\tau_1 + \frac{5}{6}\tau_2\right)x\right)$$

$$+a_0+\tfrac{1}{2}a_1-\tfrac{1}{4}a_2-\tfrac{1}{4}\tau_1-\tfrac{1}{4}\tau_2=0 \qquad (17)$$

Now, collocate (17) at the equidistant points, that is, $x_k=\left(\frac{b-a}{n+1}\right)k$ with $n=2,\ k=1,2,3$, to obtain the following systems of equations:

$$A_1 := \frac{1}{332640}\frac{1}{\pi^{\frac{3}{2}}}(48510a_1\pi^{\frac{3}{2}}+30492a_2\pi^{\frac{3}{2}}+30492\tau_1\pi^{\frac{3}{2}}+30492\tau_2\pi^{\frac{3}{2}}+9240\pi^{\frac{3}{2}}$$
$$-392832a_2\pi-392832\tau_1\pi-392832\tau_2\pi-133056\pi a_1-12672\pi)=0$$

$$A_2 := \frac{1}{332640}\frac{1}{\pi^{\frac{3}{2}}}(48510a_0\pi^{\frac{3}{2}}+28413a_2\pi^{\frac{3}{2}}+28413\tau_1\pi^{\frac{3}{2}}+28413\tau_2\pi^{\frac{3}{2}}+42966a_1\pi^{\frac{3}{2}}$$
$$+8085\pi^{\frac{3}{2}}-133056\pi a_0-228096\pi a_1-24992\pi+332640\sqrt{\pi}a_1+969960\sqrt{\pi}a_2$$
$$+969960\sqrt{\pi}\tau_1+9699600\sqrt{\pi}\tau_2-406208\pi a_2-406208\pi\tau_1-406208\pi\tau_2$$
$$+73920\sqrt{\pi})=0$$

$$A_3 := \frac{1}{332640}\frac{1}{\pi^{\frac{3}{2}}}(30492a_0\pi^{\frac{3}{2}}+28413a_1\pi^{\frac{3}{2}}+22572\tau_1\pi^{\frac{3}{2}}22572\tau_2\pi^{\frac{3}{2}}+5082\pi^{\frac{3}{2}}$$
$$-392832\pi a_0-412928\pi a_2-4537\pi+2808960\sqrt{\pi}a_2-960960\sqrt{\pi}a_1$$
$$+2808960\sqrt{\pi}\tau_1+280890\sqrt{\pi}\tau_2-412928\pi\tau_1--412928\pi\tau_2+147840\sqrt{\pi})=0$$

Using the initial condition on $u(0)=u'(0)=0$, on (17), we obtain,

$$A_4 := a_0+\frac{1}{2}a_1-\frac{1}{2}a_2-\frac{1}{2}\tau_1-\frac{1}{2}\tau_2=0$$

$$A_5 := \frac{1}{2}a_1+a_2+\tau_1+\tau_2=0$$

Solving the above equations via MAPLE 18, we obtain values

$a_0 = 2.600000, a_1 = -4.000000, a_2 = 2.400000, a_3 = 0.000000, \tau_1 = 0.000000,$
$\tau_2 = -0.000000$

Substituting into (17), we have the approximate solution to (10) as

$$u(x)=x^2.$$

The computational results are shown in Table 2.

Table 1: Comparison of the result between the exact and the approximate solution for case 1 and example 4.1; Case 1

X	Exact	Approximate Solution (MNPS)	Approximate Solution (CPS)	Error (MNPS & CPS)
0.01	-0.0099000	-0.0099000	-0.0099000	0.0000e+00
0.02	-0.0196000	-0.0196000	-0.0196000	0.0000e+00
0.03	-0.0291000	-0.0291000	-0.0291000	0.0000e+00
0.04	-0.0384000	-0.0384000	-0.0384000	0.0000e+00
0.05	-0.0475000	-0.0475000	-0.0475000	0.0000e+00
0.06	-0.0564000	-0.0564000	-0.0564000	0.0000e+00
0.07	-0.0651000	-0.0651000	-0.0651000	0.0000e+00
0.08	-0.0736000	-0.0736000	-0.0736000	0.0000e+00
0.09	-0.0819000	-0.0819000	-0.0819000	0.0000e+00
0.10	-0.0900000	-0.0900000	-0.0900000	0.0000e+00

Table 2: Comparison of result between the exact and the approximate solution for case 1 and example 4.2; Case 2

X	Exact Solution	Approximate Solution (MNPS)	Approximate Solution (CPS)	Error (MNPS & CPS)
0.01	0.0001000	0.0001000	0.0001000	0.0000e+00
0.02	0.0004000	0.0004000	0.0004000	0.0000e+00
0.03	0.0009000	0.0009000	0.0009000	0.0000e+00
0.04	0.0016000	0.0016000	0.0016000	0.0000e+00
0.05	0.0025000	0.0025000	0.0025000	0.0000e+00
0.06	0.0036000	0.0036000	0.0036000	0.0000e+00
0.07	0.0049000	0.0049000	0.0049000	0.0000e+00
0.08	0.0064000	0.0064000	0.0064000	0.0000e+00
0.09	0.0081000	0.0081000	0.0081000	0.0000e+00
0.10	0.0100000	0.0100000	0.0100000	0.0000e+00

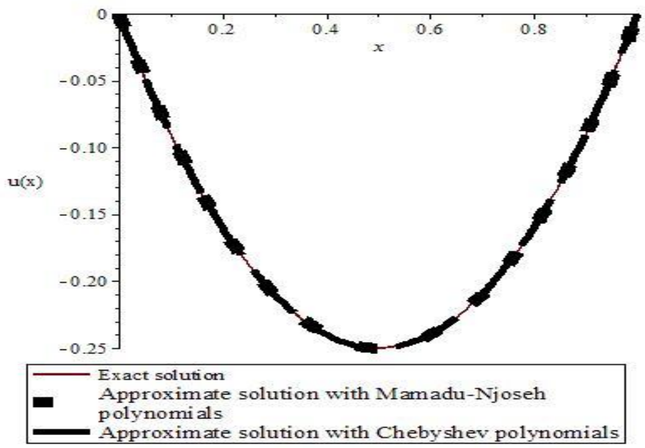

Figure 1: Comparison for Case 1

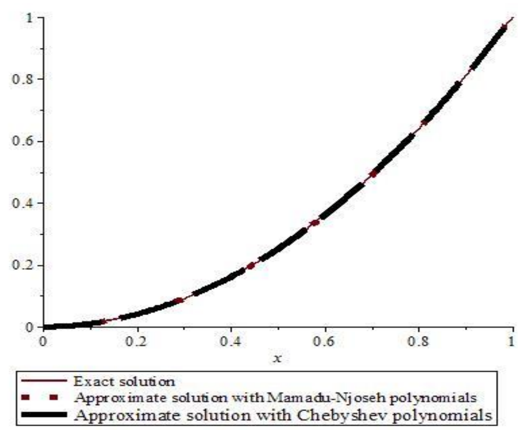

Figure 2: Comparison for Case 2

5. Discussion of Results

The accuracy of the Tau-Collocation approach for the solution of the fractional integro-differential equation has been viewed from two numerical examples. Both Mamadu-Njoseh and Chebyshev orthogonal polynomials were applied as basis functions to seek the approximate solutions of the numerical examples considered. The resulting numerical evidence for both examples considered with Mamadu-Njoseh and Chebyshev basis functions shows absolute convergence as shown in **Figures 1** and **2**. This observation suggests that the Mamadu-Njoseh and Chebyshev orthogonal basis functions exhibit the same rate of convergence as noted in Mamadu and Njoseh (2016b) and Njoseh and Mamadu (2016d).

6. Conclusion

This study considered a numerical method for solving fractional Fredholm integro-differential equations. It was established that the method is an effective solver for integro-differential equations and is highly accurate. It is also evident that the method offers several advantages which include cost-effectiveness, as no extra interpolation is required in other to achieve several outputs, ease of implementation, easy to program and excellent rate of convergence

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