FRACTIONAL TRANSMISSION MODEL FOR ZIKA VIRUS

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Abstract

In this work, we studied the dynamics of a Zika virus model within the framework of the Caputo fractional derivative. Using a fixed-point approach, we establish conditions for which the considered fractional model admits a unique system of solutions. The two-step Adams-Bashforth numerical scheme incorporating the fractional order parameter index σ is then used to furnish numerical simulations demonstrating the behaviour of the model state variables with respect to distinct values of the fractional order parameter index.

As the value of σ increases from 0.7 to 1, there is decrease in the number of susceptible individuals and then a gradual increase after some time t, until it steadies at equilibrium.

It was also observed that as the value of the fractional order parameter increases from 0.7 to 1 the number of exposed and infected individual decreases while the number of recovered individual increases after some time. Furthermore as the value of susceptible and exposed vector decreases, the number of infected vector increases and then decreases after some time.

Keywords: Fixed-point; Numerical scheme; Fractional derivatives; Fractional order parameter; Zika virus.

Introduction

Zika virus was named after the Zika forest in Uganda where it was first discovered over 70 years ago by some researcher who were working on YFVD and was first isolated in a rhesus monkey. The virus belongs to the genus Flavi virus family with about 53 different species and has no specified antiviral drug or vaccine (Dick et al., 1952). Nigeria discovered and reported her first human cases in year 1954.

Zika virus disease is caused by bites of infected female Aedes mosquitoes *"Aedes aegypti "*which are phylogenetically related to the ones that cause mosquito borne flavi viruses. This virus can be found in Blood, Urine, Saliva, semen, breast milk.

It is well documented that the transmission of Zika virus is possible via three main route, namely, between human to human via sexual interaction, transplacental transmission between an infected mother and the new born child, unscreened blood

transfusion etc, human to vector and vector to human. This allows us to take into account transmissions arising from both human-to-human interactions as well as interactions between vector and human.

Prevention of Zika virus involves deliberate measures which include prevention of arbo viral infection by using mosquito treated nets, wearing long gear covering body, insecticides, repellents, or bird nets. Efforts towards having a vaccine to prevent Zika virus are currently ongoing (Barrett, 2018).

Several authors have studied different mathematical models describing the spread of Zika virus within the framework of systems of ordinary differential equations within the framework of classical (or integer-order) and fractional derivatives, however, models with integer-order derivatives do not adequately account for hereditary and memory effects associated with many biological processes (Rezapour et. al., 2020), the fractional order operators incorporate hereditary properties and provides good description of the memory effects associated with many physical systems.

In this article, we employ a SEIR-type epidemic modeling framework comprising of two sub-populations, namely, human and mosquito vector sub-populations to investigate the dynamics of a fractional Zika virus model with seven epidemiological compartments consisting of susceptible, exposed, infectious and recovered humans as well as susceptible, exposed and infectious mosquitoes within the framework of the Caputo fractional derivative. For each of the mentioned sub-populations, a compartmental model is constructed to simulate Zika virus transmission while an interaction between both sub-populations occurs through mosquito bites. We incorporate a linear incidence term for both human to human and vector to human transmissions.

2 Preliminaries

In the present section, we present some fundamental definitions and properties on fractional differential and integral operators related to the Caputo type.

Definition 1. (Podlubny, 1999; Caputo, 1967) The Riemann-Liouville fractional integral of order $\sigma$of a function $g \in C_{\mu},\mu \geq$ - 1 is defined as

$$I_t^{\sigma}[g(t)] = \begin{cases} g(t), & \sigma = 0, t > 0, \\ \dfrac{1}{\Gamma(\sigma)} \displaystyle\int_0^t (t-\tau)^{\sigma-1} g(\tau) d\tau, & \sigma > 0, t > 0, \end{cases} \tag{2.1}$$

where 0 <σ<1.

Definition 2. (Podlubny, 1999; Caputo, 1967) The Caputo fractional derivative of order $\sigma$of a function $g \in C_{-1}^{m}, m \in \mathbb{N} \cup \{0\}$ is defined as

$$ {}_0^C D_t^{\sigma}[g(t)] = \begin{cases} g^{(m)}(t) \coloneqq \dfrac{d^m g(t)}{dt^m}, & \sigma = m, \\ \dfrac{1}{\Gamma(m-\sigma)} \displaystyle\int_0^t g^{(m)}(t)(t-\tau)^{m-\sigma-1} d\tau, & m-1 < \sigma < m. \end{cases} \tag{2.2} $$

Lemma 1. (Atangana and Owolabi, 2018) Let $\sigma \in (0,1)\ and\ \mathcal{H} \in C([0,T], \mathbb{R}_+)$ be a nonlinear function. Then the fractional IVP in Caputo derivatives

$$ \begin{cases} {}_0^{\mathbb{C}} D_t^{\sigma} \theta(t) = \mathcal{H}\big(t, \theta(t)\big), t \in [0,T], \\ \theta(0) = \theta_0, \end{cases} \tag{2.3} $$

has a unique solution given as

$$ \theta(t) = \theta_0 + \frac{1}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} \mathcal{H}\big(s, \theta(s)\big) ds. \tag{2.4} $$

3

MODEL FORMULATION

In the present consideration, we consider the fractional version of the model proposed by Ali et. al. (2022). the total human population denoted by $N_h(t)$ is subdivided into four independent epidemiological compartments, namely, susceptible humans $S_h(t)$, exposed humans $E_h(t)$, infected humans $I_h(t)$, and recovered humans $R_h(t)$, such that $N_h(t) = S_h(t) + E_h(t) + I_h(t) + R_h(t)$while the entire vector population denoted by $N_v(t)$ is subdivided into three independent epidemiological compartments, namely, susceptible vectors $S_v(t)$, exposed vectors $E_v(t)$ and infected mosquito $I_v(t)$ such that$N_v(t) = S_v(t) + E_v(t) + I_v(t)$.

Susceptible humans and mosquitoes are recruited into the susceptible compartments $S_h$and $S_v$at rates $\Pi_h$and Π_v, respectively. We represent by $\lambda S_h = (\lambda_1 + \lambda_2)S_h$the incidence rate of infection in the human population where $\lambda_1 = \beta_h I_v$ is the rate at which susceptible individuals acquire infection due to effective contact with an infected vector and $\lambda_2=\rho\beta_h I_h$is the rate at which susceptible individuals acquire infection due to sexual interaction with infected individuals. Here, $\beta_h$is the effective contact rate between susceptible humans and infected mosquitoes while $\rho$is a modification parameter that accounts for the relative infectiousness of individuals in the I_h relative to those in the I_v compartment. Similarly, we represent by $\beta_v I_h S_v$ the incidence rate of the susceptible vector population where $\beta_v$denotes the transmission rate from infected humans to susceptible mosquito. The disease induced mortality rate is denoted by δ. Natural mortality rates due for the human and vector subpopulations are denoted by μ_h and μ_v respectively. Lastly, γ and $\tau$are the natural and treatment rates. Following the above description for the interrelationship between compartments, we arrive at the following coupled system of nonlinear ordinary differential equations describing the Zika virus dynamics:

$$\begin{cases} \frac{S_h(t)}{dt} = \Pi_h - \beta_h S_h(I_v + \rho I_h) - \mu_h S_h \\ \frac{E_h(t)}{dt} = \beta_h S_h(I_v + \rho I_h) - (\mu_h + \chi)E_h \\ \frac{I_h(t)}{dt} = \chi E_h - (\mu_h + \gamma + \tau)I_h \\ \frac{R_h(t)}{dt} = \gamma I_h - \mu_h R_h \\ \frac{S_v(t)}{dt} = \Pi_v - \beta_v S_v I_v - \mu_v S_v \\ \frac{E_v(t)}{dt} = \beta_v S_v I_v - (\mu_v + \delta)E_v \\ \frac{I_v(t)}{dt} = \delta E_v - \mu_v I_v, \end{cases} \tag{3.1}$$

With associated initial conditions

$Sh(0) = Sh0, Eh(0) = Eh0, Ih(0) = Ih0, Rh(0) = Rh0, Sv(0) = Sv0, Ev(0) = Ev0, Iv(0) = Iv0$.

Next, we extend the classical model (3.2) to a fractional model by incorporating the time-fractional derivative in place of the classical ordinary derivative for each equation in the system. We will consider the fractional model in the Caputo sense, namely,

$$\begin{cases} {}_0^{\mathbb{C}}D_t^{\sigma} S_h(t) = \Pi_h - \beta_h S_h(I_v + \rho I_h) - \mu_h S_h, \\ {}_0^{\mathbb{C}}D_t^{\sigma} E_h(t) = \beta_h S_h(I_v + \rho I_h) - (\mu_h + \chi)E_h, \\ {}_0^{\mathbb{C}}D_t^{\sigma} I_h(t) = \chi E_h - (\mu_h + \gamma + \tau)I_h, \\ {}_0^{\mathbb{C}}D_t^{\sigma} R_h(t) = \gamma I_h - \mu_h R_h, \\ {}_0^{\mathbb{C}}D_t^{\sigma} S_v(t) = \Pi_v - \beta_v S_v I_h - \mu_h S_h, \\ {}_0^{\mathbb{C}}D_t^{\sigma} E_v(t) = \beta_v S_v I_h - (\mu_h + \delta)E_v, \\ {}_0^{\mathbb{C}}D_t^{\sigma} I_v(t) = \delta E_v - \mu_v I_v, \end{cases} \tag{3.2}$$

with associated non-negative initial conditions

$$S_h(0) = S_h, E_h(0) = E_{h0}, I_h(0) = I_{h0}, R_h(0) = R_{h0}, \quad S_v(0) = S_{v0}, E_v(0) = E_{v0}, I_v(0) = I_{v0} \tag{3.3}$$

${}_0^{\mathbb{C}}D_t^{\sigma}$ represents the fractional differential operator in the Caputo sense with $0 < \sigma \leq 1$ being the fractional parameter index.

4 Existence and Uniqueness of Solutions

Under certain conditions, the existence and uniqueness of solutions for the considered model with respect to the Caputo fractional derivative can be investigated via a fixed-point technique. For the sake of convenience in our subsequent investigations, we make the following notations for the right-hand terms appearing in (3.2).

$$\begin{cases} \mathbb{F}_1\big(t, S_h(t)\big) = \Pi_h - \beta_h S_h(I_v + \rho I_h) - \mu_h S_h, \\ \mathbb{F}_2\big(t, E_h(t)\big) = \beta_h E_h(I_v + \rho I_h) - (\mu_h + \chi)E_h, \\ \mathbb{F}_3\big(t, I_h(t)\big) = \chi E_h - (\mu_h + \gamma + \tau)I_h, \\ \mathbb{F}_4\big(t, R_h(t)\big) = \gamma I_h - \mu_h R_h, \\ \mathbb{F}_5\big(t, S_v(t)\big) = \Pi_v - \beta_v S_v I_h - \mu_h S_h, \\ \mathbb{F}_6\big(t, E_v(t)\big) = \beta_v E_v I_h - (\mu_h + \delta)E_v, \\ \mathbb{F}_7\big(t, I_v(t)\big) = \delta E_v - \mu_v I_v, \end{cases} \tag{4.1}$$

In view of Lemma 1, applying the Riemann-Liouville integral operator on both sides of each equation in (3.2) yields the following equivalent system of fractional integral equations:

$$\begin{cases} S_h(t) = S_h(0) + \dfrac{1}{\Gamma(\sigma)}\displaystyle\int_0^t (t-\tau)^{\sigma-1}\mathbb{F}_1(\tau, S_h\tau)d\tau, \\ E_h(t) = E_h(0) + \dfrac{1}{\Gamma(\sigma)}\displaystyle\int_0^t (t-\tau)^{\sigma-1}\mathbb{F}_2\big(\tau, E_h(\tau)\big)d\tau, \\ I_h(t) = I_h(0) + \dfrac{1}{\Gamma(\sigma)}\displaystyle\int_0^t (t-\tau)^{\sigma-1}\mathbb{F}_3\big(r, I_h(\tau)\big)d\tau, \\ R_h(t) = R_h(0) + \dfrac{1}{\Gamma(\sigma)}\displaystyle\int_0^t (t-\tau)^{\sigma-1}\mathbb{F}_4\big(\tau, R_h(r)\big)d\tau, \\ S_v(t) = S_v(0) + \dfrac{1}{\Gamma(\sigma)}\displaystyle\int_0^t (t-\tau)^{\sigma-1}\mathbb{F}_5\big(\tau, S_v(\tau)\big)d\tau, \\ E_v(t) = E_v(0) + \dfrac{1}{\Gamma(\sigma)}\displaystyle\int_0^t (t-\tau)^{\sigma-1}\mathbb{F}_6\big(\tau, E_v(\tau)\big)d\tau, \\ I_v(t) = I_v(0) + \dfrac{1}{\Gamma(\sigma)}\displaystyle\int_0^t (t-\tau)^{\sigma-1}\mathbb{F}_7\big(\tau, I_v(\tau)\big)d\tau, \end{cases} \tag{4.2}$$

Next, we show that the nonlinear functions $\mathbb{F}_i$(i= 1, 2, · · · 7) defined in (4.2) satisfy the Lipschitz condition. To this end, we first consider

$$\mathbb{F}_1\big(t, S_h(t)\big) = \Pi_h - \beta_h S_h(I_v + \rho I_h) - \mu_h S_h.$$

Let $I_h$and $I_v$be two bounded functions on $t \in [0, T]$ such that $\|I_h(t)\| \le \tau_1$ and $\|I_v(t)\| \le \tau_2$ where $\|\cdot\|$ denotes the maximum norm. Then for any two functions $S_t^*(t)$ and $S_t^{**}(t)$, we have

$$\begin{aligned} \big\|\mathbb{F}_1\big(t, S_h^*(t)\big) - \mathbb{F}_1\big(t, S_t^{**}(t)\big)\big\| &= \big\|-\beta_h\big(I_v(t) + \rho I_h(t)\big)\big(S_t^*(t) - S_h^{**}(t)\big) - \mu_h\big(S_h^*(t) - S_h^{**}(t)\big)\big\| \\ &\le [\beta_h(\|I_v(t)\| + \rho\|I_h(t)\|) + \mu_h]\|S_h^*(t) - S_h^{**}(t)\| \\ &\le [\beta_h(\tau_2 + \rho\tau_1) + \mu_h]\|S_h^*(t) - S_h^{**}(t)\|. \end{aligned}$$

This implies

$$\big\|\mathbb{F}_1\big(t, S_h^*(t)\big) - \mathbb{F}_1\big(t, S_t^{**}(t)\big)\big\| \le \mathcal{K}_1\|S_h^*(t) - S_h^{**}(t)\| \tag{4.3}$$

where $\mathcal{K}_1 = \beta_h(\tau_2 + \rho\tau_1) + \mu_h$. It follows that, $\mathbb{F}_1\big(t, S_h(t)\big)$ satisfies the Lipschitz condition. Moreover, is a contraction if $0 \le \beta_h(\tau_2 + \rho\tau_1) + \mu_h < 1$ In line with the argument leading to (4.3), we can also find constants $\mathcal{K}_2 \coloneqq \mu_h + \chi, \mathcal{K}_3 \coloneqq \mu_h + \gamma + \tau, \mathcal{K}_4 \coloneqq$

$\mu_h, \mathcal{K}_5 := \beta_v\tau_1 + \mu_v, \mathcal{K}_6 := \delta + \mu_v, and\ \mathcal{K}_7 := \mu_v$such that the remaining functions $\mathbb{F}_i$(*i*= 2, 3,..., 7) satisfy

$$\begin{aligned}
\left\|\mathbb{F}_2\left(t, E_h^*(t)\right) - \mathbb{F}_2\left(t, E_h^{**}(t)\right)\right\| &\leq \mathcal{K}_2\|E_h^*(t) - E_h^{**}(t)\|,\\
\left\|\mathbb{F}_3\left(t, I_h^*(t)\right) - \mathbb{F}_3\left(t, I_h^{**}(t)\right)\right\| &\leq \mathcal{K}_3\|I_h^*(t) - I_h^{**}(t)\|,\\
\left\|\mathbb{F}_4\left(t, R_h^*(t)\right) - \mathbb{F}_4\left(t, R_h^{**}(t)\right)\right\| &\leq \mathcal{K}_4\|R_h^*(t) - I_h^{**}(t)\|,\\
\left\|\mathbb{F}_5\left(t, S_v^*(t)\right) - \mathbb{F}_5\left(t, S_v^{**}(t)\right)\right\| &\leq \mathcal{K}_5\|S_v^*(t) - S_v^{**}(t)\|,\\
\left\|\mathbb{F}_6\left(t, E_v^*(t)\right) - \mathbb{F}_6\left(t, E_v^{**}(t)\right)\right\| &\leq \mathcal{K}_6\|E_v^*(t) - E_v^{**}(t)\|,\\
\left\|\mathbb{F}_7\left(t, I_v^*(t)\right) - \mathbb{F}_7\left(t, I_v^{**}(t)\right)\right\| &\leq \mathcal{K}_7\|I_v^*(t) - I_v^{**}(t)\|,
\end{aligned}$$

Furthermore, these functions are contractions if $0 \leq \mathcal{K}_i < 1$ (*i*= 2.3. $\cdots$, 7).

Next, in view of (4.1) we rewrite the fractional Zika virus model (3.2) as

$$\begin{cases} {}_0^{\mathbb{C}}D_t^{\sigma}\mathbf{X}(t) = g(t, \mathbf{X}(t)), 0 < t < T < \infty, \\ \mathbf{X}(0) = \mathbf{X}_{0,} \end{cases} \quad (4.4)$$

Where $\mathbf{X}(t)$, $\mathbf{X}(0)$ and $g(t, \mathbf{X}(t))$ are vector functions defined as

$$\begin{cases} \mathbf{X}(t) = \begin{pmatrix} S_h(t) \\ E_h(t) \\ I_h(t) \\ R_h(t) \\ S_v(t) \\ E_v(t) \\ I_v(t) \end{pmatrix}, \quad \mathbf{X}(0) = \begin{pmatrix} S_h(0) \\ E_h(0) \\ I_h(0) \\ R_h(0) \\ S_v(0) \\ E_v(0) \\ I_v(0) \end{pmatrix}, \\ g(t, \mathbf{X}(t)) = \begin{pmatrix} \mathbb{F}_1(t, S_h(t)) \\ \mathbb{F}_2(t, E_h(t)) \\ \mathbb{F}_3(t, I_h(t)) \\ \mathbb{F}_4(t, R_h(t)) \\ \mathbb{F}_5(t, S_v(t)) \\ \mathbb{F}_6(t, E_v(t)) \\ \mathbb{F}_7(t, I_v(t)) \end{pmatrix} = \begin{pmatrix} \Pi_h - \beta_h S_h(I_v + \rho I_h) - \mu_h S_h \\ \beta_h S_h(I_v + \rho I_h) - (\mu_h + \chi)E_h \\ \chi E_h - (\mu_h + \gamma + \tau)I_h \\ \gamma I_h - \mu_h R_h \\ \Pi_v - \beta_v S_v I_h - \mu_v S_v \\ \beta_v S_v I_h - (\mu_v + \delta)E_v \\ \delta E_v - \mu_v I_v \end{pmatrix}. \end{cases}$$

Then the problem of investigating the existence and uniqueness of solutions for the considered fractional Zika virus model (3.2) is equivalent to that of investigating the existence and uniqueness of solutions to the fractional IVP (4.4). In this direction, we establish the following theorem which is adapted from (Lyons, 2017).

Theorem 4.1 *Assume that the nonlinear functions* $\mathbb{F}_i(t)$(*i*= 1, 2, $\cdots$,7) *satisfy the Lipschitz condition and that there exist constants* M_i>0 (*i*= 1, 2, $\cdots$,7) *such that* $\|\mathbb{F}_i(t,\cdot)\| \leq M_i$*hold on the rectangle* $\mathbb{B} = \{(t, X): |t| \leq a, |X - X_0| \leq b\}$*containing thepoint* (0, ***X***0). *Then the*

fractional IVP (4.4) admits a unique solution on the interval $\mathcal{J} \coloneqq \{t: |t| \leq h\}$ *where* $h = \max\{h_1, h_2, \dots, h_r\} = min\left\{a, \left(\frac{b}{M}\Gamma(\sigma+1)\right)^{\frac{1}{\sigma}}\right\}$ *and* $a \coloneqq \max(a_1, a_2, \dots, a_7), b \coloneqq \max\{b_1, b_2, \dots, b_7\}$ *and* $M \coloneqq \max\{M_1, M_2, \dots, M_7\}$*are positive constants.*

Proof: In view of (4.2), we have the following equivalent Voltera-type integral equation:

$$\mathbf{X}(t) = \mathbf{X}_0 + \frac{1}{\Gamma(\sigma)}\int_0^t (t-\tau)^{\sigma-1} g\big(\tau, \mathbf{X}(\tau)\big)d\tau. \tag{4.5}$$

Existence of solutions: We will consider only the interval [0, h] as a similar argument also holds in the interval [−h, 0]. The proof for the existence of solutions is established by constructing a sequence $\{\mathbf{X}_k(t)\}$ $(k = 1,2,3, \dots)$of successive approximations (Picard's iterates)

$$\mathbf{X}_0(t) = \mathbf{X}_0, \mathbf{X}_{k+1}(t) = \mathbf{X}_0 + \frac{1}{\Gamma(\sigma)}\int_0^t (t-\tau)^{\sigma-1} g\big(\tau, \mathbf{X}_k(\tau)\big)d\tau, k = 0,1,2, \dots, t\epsilon[0,h], \tag{4.6}$$

which converges uniformly to a function which is a solution of the integral equation (4.5) on [0, h]. This is accomplished in the following steps:

STEP I: We show via an induction argument that the sequence of successive approximations constructed in (4.6) are properly defined for every $t\epsilon[0,h]$, and in this interval the following inequality holds

$$\begin{cases} |\mathrm{X}_k(t) - \mathrm{X}_0| \leq \dfrac{Mh^\sigma}{\Gamma(\sigma+1)} \leq b \\ |\mathrm{X}_{k+1}(t) - \mathrm{X}_k(t)| \leq \dfrac{MK^k t^{(k+1)\sigma}}{\Gamma\big((k+1)\sigma+1\big)} \end{cases} \tag{4.7}$$

Note that $\mathbf{X}_{k+1}(t)$ is well defined in the interval [0, h] if the point (t,$\mathbf{X}_k(t)$) remains in the rectangle $\mathbb{B}$for every $t \epsilon$ [0, h]. Clearly, for k = 0, $\mathbf{X}_0(t)$ is defined on [0, h] and satisfies (4.7) trivially on [0, h]. Now, assume that for $k = n \geq 1$, $\mathbf{X}_n$is defined and satisfies (4.7) on [0, h], then the point (t,$\mathbf{X}_n(t)$) remains in B for t in [0, h]. Moreover, $g(t, \mathbf{X}_n(t))$ exists as a continuous function on [0, h]. Hence, by (4.6) $\mathbf{X}_n$+1(t) is defined on [0, h] and that concludes the induction argument. Furthermore, from (4.6), the induction hypothesis and (4.7) we have

$$|\mathbf{X}_{k+1}(t) - \mathbf{X}_0| \leq \frac{1}{\Gamma(\sigma)}\int_0^t (t-\tau)^{\sigma-1}|g\big(\tau, \mathbf{X}_k(\tau)\big)|d\tau \leq \frac{Mt^\sigma}{\Gamma(\sigma+1)} \leq \frac{Mh^\sigma}{\Gamma(\sigma+1)} \leq b.$$

Hence, the property (4.7) is satisfied by $\mathbf{X}_{k+1}$ and the induction argument is complete.

STEP II: In this step, we show via an induction argument that the inequality

$$|\mathbf{X}_{k+1}(t) - \mathbf{X}_k(t)| \leq \frac{M}{\mathcal{K}} \frac{(\mathcal{K}t^{\sigma})^{k+1}}{\Gamma\big((k+1)\sigma+1\big)}. \tag{4.8}$$

Holds on $[0, h]$. For the case $k = 0$, we have

$$|\mathbf{X}_{k+1}(t) - \mathbf{X}_k(t)| = \left|\frac{1}{\Gamma(\sigma)}\right| \int_0^t (t-\tau)^{\sigma-1} |g\big(\tau, \mathbf{X}_k(\tau)\big)| d\tau$$
$$\leq \frac{1}{\Gamma(\sigma)} \int_0^t (t-r)^{\sigma-1} |g\big(\tau, \mathbf{X}_0(\tau)\big)| d\tau \leq \frac{Mt^{\sigma}}{\Gamma(\sigma+1)}.$$

Assume that (4.8) is true for $\mathcal{K} \geq 1$, that is

$$|\mathbf{X}_{k+1}(t) - \mathbf{X}_k(t)| \leq \frac{Mt^{\sigma}}{\Gamma(k_{\sigma}+1)} t^{k\sigma}$$

on$[0, h]$where $\mathcal{K}$ is the Lipschetz constant. Then

$$|\mathbf{X}_{k+1}(t) - \mathbf{X}_k(t)|$$
$$\leq \frac{1}{\Gamma(\sigma)} \int_0^t (t-r)^{\sigma-1} \big|g\big(\tau, \mathbf{X}_k(\tau)\big) - g\big(\tau, \mathbf{X}_{k-1}(\tau)\big)\big| d\tau$$
$$\leq \frac{1}{\Gamma(\sigma)} \int_0^t (t-\tau)^{\sigma-1} |\mathbf{X}_k(\tau) - \mathbf{X}_{k-1}(\tau)| d\tau$$
$$\leq \frac{M\mathcal{K}^k}{\Gamma(\sigma)\Gamma(k\sigma+1)} \int_0^t (t-\tau)^{\sigma-1} \tau^{k\sigma} d\tau.$$

Using the substitution $\tau = \omega t$, we have

$$\frac{M\mathcal{K}^k}{\Gamma(\sigma)\Gamma(k\sigma+1)} \int_0^t (t-\tau)^{\sigma-1} (\tau-0)^{k\sigma} d\tau$$
$$= \frac{M\mathcal{K}^k t^{(k+1)\sigma}}{\Gamma(\sigma)\Gamma(k\sigma+1)} \int_0^t (t-\omega)^{\sigma-1} \omega^{k\sigma} d\tau$$
$$= \frac{M\mathcal{K}^k t^{(k+1)\sigma}}{\Gamma(\sigma)\Gamma(k\sigma+1)} B(\sigma, k\sigma+1)$$
$$= \frac{M\mathcal{K}^k t^{(k+1)\sigma}}{\Gamma(\sigma)\Gamma(k\sigma+1)} \frac{M\mathcal{K}^k t^{(k+1)\sigma}}{\Gamma\big((k+1)\sigma+1\big)}$$
$$= \frac{M\mathcal{K}^k t^{(k+1)\sigma}}{\Gamma\big((k+1)\sigma+1\big)}.$$

Here $\boldsymbol{B}(.,.)$ denotes the Beta function. Hence, we have

$$|\mathbf{X}_{k+1}(t) - \mathbf{X}_k(t)| \leq \frac{M\mathcal{K}^k t^{(k+1)\sigma}}{\Gamma\big((k+1)\sigma+1\big)}$$

Therefore, the inequality (4.8) is true for all *k*.

STEP III. We show that the sequence of functions $\{\mathbf{X}_k(t)\}$ defined in (4.6) converge uniformly on [0, *h*]. To this end, we observe that the sequence of successive approximations consists of the sequence of partial sum of the series

$$\mathbf{X}_0(t) + \sum_{k=0}^{\infty} [\mathbf{X}_{k+1}(t) - \mathbf{X}_k(t)]. \tag{4.9}$$

To prove uniform convergence of $\{\mathbf{X}_k(t)\}$, it is enough to establish that the partial sums of the series (4.9) are uniformly convergent. From (4.8) we have

$$\sum_{k=0}^{\infty} |\mathbf{X}_{k+1}(t) - \mathbf{X}_k(t)| \leq \frac{M}{\mathcal{K}} \sum_{k=0}^{\infty} \frac{(\mathcal{K}t^{\sigma})^{(k+1)}}{\Gamma\big((k+1)\sigma + 1\big)}. \tag{4.10}$$

In other words, the terms in (4.9) are bounded in absolute value on the interval [0, h] by the terms in the positive series

$$\frac{M}{\mathcal{K}} \sum_{k=0}^{\infty} \frac{(\mathcal{K}t^{\sigma})^{k+1}}{\Gamma\big((k+1)\sigma + 1\big)}$$

which converges to $\frac{M}{\mathcal{K}} E_{\sigma,1}(\mathcal{K}t^{\sigma})$. Hence, by Weierstrass M−test, the series in (4.9) converges uniformly on [0, h]. Therefore, the sequence $\{\mathbf{X}_k\}$ is uniformly convergent on [0, h] to continuous function say $\mathbf{X}(t)$.

STEP IV. In this step, we show that $\mathbf{X}(t)$ is the solution of the integral equation (4.5). Since the sequence $\{\mathbf{X}_k\}$ is uniformly convergent to some continuous function $\mathbf{X}(t)$, then from (4.6) we have

$$\begin{aligned} \mathbf{X}(t) &= \lim_{k\to\infty} \mathbf{X}_{k+1}(t) \\ &= \lim_{k\to\infty} \left[X_0 + \frac{1}{\Gamma(\sigma)} \int_0^t (t-\tau)^{\sigma-1} g\big(\tau, \mathbf{X}_k(\tau)\big) d\tau \right] \\ &= X_0 + \frac{1}{\Gamma(\sigma)} \int_0^t (t-\tau)^{\sigma-1} g\big(\tau, \mathbf{X}(\tau)\big) d\tau \end{aligned} \tag{4.11}$$

Since $\mathbf{X}(t)$ satisfies the equivalent integral equation, then it must satisfy the initial value problem (4.4).

Uniqueness: To complete the proof of the theorem, we establish uniqueness of solution to the fractional IVP (4.4) with Caputo derivative by claiming that there exists another solution $\widetilde{\mathbf{X}}(t) := (\tilde{S}_h(t), \tilde{E}_h(t), \tilde{I}_h(t), \tilde{R}_h(t), \tilde{S}_v(t), \tilde{E}_v(t), \tilde{I}_v(t))^{\tau}$.

Now, consider the function $\theta(t) = |\mathbf{X}(t) - \widetilde{\mathbf{X}}(t)|$. Since both $\mathbf{X}(t)$ and $\tilde{X}(t)$ are solutions to the IVP (4.4), we see that $\theta_0 = |\mathbf{X}_0 - \widetilde{\mathbf{X}}_0| = 0$. Furthermore, from Theorem 4.1we have

$$\begin{aligned}\theta(t) &= \left|\mathbf{X}(t) - \widetilde{\mathbf{X}}(t)\right| \\ &= \left|\frac{1}{\Gamma(\sigma)}\int_0^t (t-\tau)^{\sigma-1} g\big(\tau, \mathbf{X}(\tau)\big) - g\left(\tau, \widetilde{\mathbf{X}}(\tau)\right) d\tau\right| \\ &\leq \frac{1}{\Gamma(\sigma)}\int_0^t (t-\tau)^{\sigma-1}\left|g\big(\tau, \mathbf{X}(\tau)\big) - g\left(\tau, \widetilde{\mathbf{X}}(\tau)\right)\right| d\tau \qquad (4.12) \\ &\leq \frac{\mathcal{K}}{\Gamma(\sigma)}\int_0^t (t-\tau)^{\sigma-1}\theta(\tau)d\tau.\end{aligned}$$

Since $\mathcal{K} > 0$ *and* $\theta(t) \geq 0$, the inequality in (4.12) satisfies all conditions of the Gronwall-type inequality (Denton and Vatsala, 2010). Thus, we have

$$\theta(t) \leq \theta_0 E_{\sigma,1}(\mathcal{K}t^{\sigma}) = 0$$

This proves that $X = \tilde{X}$and hence, the solution of the fractional IVP (4.4) exists on [0, *h*] and is unique. Consequently, the fractional Zika virus model (3.2) in Caputo derivative admits unique solution.

Remark 4.5*:*It is easy to see that if σ= 1, the proof Theorem 4.4 is similar to Picard's original proof for systems of ordinary differential equations with integer order derivatives.

5 Numerical Schemes

We present corresponding FAB schemes for the fractional Zika virus models (4.1) with fractional derivative in the Caputo sense. We furnish graphical visualizations comparing the behaviours of each state variables of the fractional Zika virus models (4.1) for distinct values of the fractional order parameter σ. The simulation parameter values are taken as Π_h = 0.8,Π_v= 0.08, β_h= 0.007, β_v= 0.009, μ_h= 0.0028, μ_v= 0.071, ρ= 0.05, X= 0.7, γ= 0.05, τ= 0.08, δ= 0.5 (Bonyah et al, 2017) while the initial values used are S_{h0} = 100, E_{h0} = 10, I_{h0} = 30, R_{h0} = 20, S_{v0} = 10, E_{v0} = 50, I_{v0} = 10.

5.1 FAB scheme for model in Caputo derivative

Using the fundamental theorem of integral calculus, we obtain the following corresponding nonlinear fractional Volterrra-type integral equation

$$S_h(t) - S_h(0) = \frac{1}{\Gamma(\sigma)}\int_0^t (t-\tau)^{\sigma-1}\mathbb{F}_1\big(\tau, S_h(\tau)\big)d\tau. \qquad (5.1)$$

for the S_h–equation (4.5) in Caputo derivative. At $t = t_{k+1}$and $t = t_k$, k = 0, 1, 2, …, (5.1) can be read as

$$S_h(t_{k+1}) - S_h(0) = \frac{1}{\Gamma(\sigma)}\int_0^{t_{k+1}} (t_{k+1}-t)^{\sigma-1}\mathbb{F}_1\big(t, S_h(t)\big)dt$$

and

$$S_h(t_k) - S_h(0) = \frac{1}{\Gamma(\sigma)} \int_0^{t_k} (t_k - t)^{\sigma-1} \mathbb{F}_1\big(t, S_h(t)\big) dt.$$

respectively. We easily see that,

$$S_h(t_{k+1}) - S_h(t_k) = \mathrm{X}_{\sigma,1} - \mathrm{X}_{\sigma,2}, \tag{5.2}$$

where:

$$\mathrm{X}_{\sigma,1} = \frac{1}{\Gamma(\sigma)} \int_0^{t_{k+1}} (t_{k+1} - t)^{\sigma-1} \mathbb{F}_1\big(t, S_h(t)\big) dt, \tag{5.3}$$

and

$$\mathrm{X}_{\sigma,2} = \frac{1}{\Gamma(\sigma)} \int_0^{t_k} (t_k - t)^{\sigma-1} \mathbb{F}_1\big(t, S_h(t)\big) dt, \tag{5,4}$$

Over the interval $[t_k, t_{k+1}]$, the function $\mathrm{F}_1(t, S_h(t))$ can be approximated by the two-point Lagrange interpolation polynomial of the form

$$\begin{aligned}\mathbb{F}_1\big(t, S_h(t)\big) &\simeq \frac{t - t_{k-1}}{t_k - t_{k-1}} \mathbb{F}_1\big(t_k, S_h(t_k)\big) + \frac{t - t_k}{t_{k-1} - t_k} \mathbb{F}_1\big(t_{k-1}, S_h(t_{k-1})\big) \\ &= \frac{t - t_{k-1}}{h} \mathbb{F}_1\big(t_k, S_h(t_k)\big) - \frac{t - t_k}{h} \mathbb{F}_1\big(t_{k-1}, S_h(t_{k-1})\big),\end{aligned}$$

Where $h = t_k - t_{k-1}$ is the step-size. Substituting (5.5) into the first and second integrals in (5.3) yield

$$\mathrm{X}_{\sigma,1} = \frac{\mathbb{F}_1(t_k, S_h(t_k))}{h\Gamma(\sigma)} \left[\frac{2ht_{k+1}^{\sigma}}{\sigma} - \frac{t_{k+1}^{\sigma+1}}{\sigma+1}\right] - \frac{\mathbb{F}_1\big(t_{k-1}, S_h(t_{k-1})\big)}{h\Gamma(\sigma)} \left[\frac{ht_{k+1}^{\sigma}}{\sigma} - \frac{t_{k+1}^{\sigma+1}}{\sigma+1}\right], \tag{5.5}$$

And

$$\mathrm{X}_{\sigma,2} = \frac{\mathbb{F}_1(t_k, S_h(t_k))}{h\Gamma(\sigma)} \left[\frac{ht_k^{\sigma}}{\sigma} - \frac{t_k^{\sigma+1}}{\sigma+1}\right] - \frac{\mathbb{F}_1\big(t_k, S_h(t_{k-1})\big)}{h\Gamma(\sigma)} \frac{t_k^{\sigma+1}}{\sigma+1}, \tag{5.6}$$

respectively, after some manipulations. By inserting (5.5) and (5.6) into (5.2), we obtain

$$\begin{aligned}\mathrm{S}_h(t_{k+1}) = S_h(t_k) &+ \frac{\mathbb{F}_1(t_k, S_h(t_k))}{h\Gamma(\sigma)} \left[\frac{2ht_{k+1}^{\sigma}}{\sigma} - \frac{t_{k+1}^{\sigma+1}}{\sigma+1} - \frac{ht_k^{\sigma}}{\sigma} + \frac{t_k^{\sigma+1}}{\sigma+1}\right] \\ &- \frac{\mathbb{F}_1\big(t_{k-1}, S_h(t_{k-1})\big)}{h\Gamma(\sigma)} \left[\frac{ht_{k+1}^{\sigma}}{\sigma} - \frac{t_{k+1}^{\sigma+1}}{\sigma+1} + \frac{t_k^{\sigma+1}}{\sigma+1}\right]\end{aligned} \tag{5.7}$$

as the final two-step FAB scheme for the S_h–equation (3.2) with Caputo derivative. In the same way, we can obtain a similar scheme for each of the remaining equations in (3.2). In particular, we have

$$\begin{aligned}E_h(t_{k+1}) = E_h(t_k) &+ \frac{\mathbb{F}_1(t_k, E_h(t_k))}{h\Gamma(\sigma)} \left[\frac{2ht_{k+1}^{\sigma}}{\sigma} - \frac{t_{k+1}^{\sigma+1}}{\sigma+1} - \frac{ht_k^{\sigma}}{\sigma} + \frac{t_k^{\sigma+1}}{\sigma+1}\right] \\ &- \frac{\mathbb{F}_1\big(t_{k-1}, E_h(t_{k-1})\big)}{h\Gamma(\sigma)} \left[\frac{ht_{k+1}^{\sigma}}{\sigma} - \frac{t_{k+1}^{\sigma+1}}{\sigma+1} + \frac{t_k^{\sigma+1}}{\sigma+1}\right], \\ I_h(t_{k+1}) = I_h(t_k) &+ \frac{\mathbb{F}_1(t_k, I_h(t_k))}{h\Gamma(\sigma)} \left[\frac{2ht_{k+1}^{\sigma}}{\sigma} - \frac{t_{k+1}^{\sigma+1}}{\sigma+1} - \frac{ht_k^{\sigma}}{\sigma} + \frac{t_k^{\sigma+1}}{\sigma+1}\right] \\ &- \frac{\mathbb{F}_1\big(t_{k-1}, I_h(t_{k-1})\big)}{h\Gamma(\sigma)} \left[\frac{ht_{k+1}^{\sigma}}{\sigma} - \frac{t_{k+1}^{\sigma+1}}{\sigma+1} + \frac{t_k^{\sigma+1}}{\sigma+1}\right],\end{aligned}$$

$$R_h(t_{k+1}) = R_h(t_k) + \frac{\mathbb{F}_1(t_k, R_h(t_k))}{h\Gamma(\sigma)}\left[\frac{2ht_{k+1}^{\sigma}}{\sigma} - \frac{t_{k+1}^{\sigma+1}}{\sigma+1} - \frac{ht_k^{\sigma}}{\sigma} + \frac{t_k^{\sigma+1}}{\sigma+1}\right] - \frac{\mathbb{F}_1\left(t_{k-1}, R_h(t_{k-1})\right)}{h\Gamma(\sigma)}\left[\frac{ht_{k+1}^{\sigma}}{\sigma} - \frac{t_{k+1}^{\sigma+1}}{\sigma+1} + \frac{t_k^{\sigma+1}}{\sigma+1}\right],$$

$$S_v(t_{k+1}) = S_v(t_k) + \frac{\mathbb{F}_1(t_k, S_v(t_k))}{h\Gamma(\sigma)}\left[\frac{2ht_{k+1}^{\sigma}}{\sigma} - \frac{t_{k+1}^{\sigma+1}}{\sigma+1} - \frac{ht_k^{\sigma}}{\sigma} + \frac{t_k^{\sigma+1}}{\sigma+1}\right] - \frac{\mathbb{F}_1\left(t_{k-1}, S_v(t_{k-1})\right)}{h\Gamma(\sigma)}\left[\frac{ht_{k+1}^{\sigma}}{\sigma} - \frac{t_{k+1}^{\sigma+1}}{\sigma+1} + \frac{t_k^{\sigma+1}}{\sigma+1}\right],$$

$$E_v(t_{k+1}) = E_v(t_k) + \frac{\mathbb{F}_1(t_k, E_v(t_k))}{h\Gamma(\sigma)}\left[\frac{2ht_{k+1}^{\sigma}}{\sigma} - \frac{t_{k+1}^{\sigma+1}}{\sigma+1} - \frac{ht_k^{\sigma}}{\sigma} + \frac{t_k^{\sigma+1}}{\sigma+1}\right] - \frac{\mathbb{F}_1\left(t_{k-1}, E_v(t_{k-1})\right)}{h\Gamma(\sigma)}\left[\frac{ht_{k+1}^{\sigma}}{\sigma} - \frac{t_{k+1}^{\sigma+1}}{\sigma+1} + \frac{t_k^{\sigma+1}}{\sigma+1}\right],$$

$$I_v(t_{k+1}) = I_v(t_k) + \frac{\mathbb{F}_1(t_k, I_v(t_k))}{h\Gamma(\sigma)}\left[\frac{2ht_{k+1}^{\sigma}}{\sigma} - \frac{t_{k+1}^{\sigma+1}}{\sigma+1} - \frac{ht_k^{\sigma}}{\sigma} + \frac{t_k^{\sigma+1}}{\sigma+1}\right] - \frac{\mathbb{F}_1\left(t_{k-1}, I_v(t_{k-1})\right)}{h\Gamma(\sigma)}\left[\frac{ht_{k+1}^{\sigma}}{\sigma} - \frac{t_{k+1}^{\sigma+1}}{\sigma+1} + \frac{t_k^{\sigma+1}}{\sigma+1}\right]. \quad (5.8)$$

as the two-step FAB scheme for the fractional Zika virus model (3.2) with Caputo derivative.

5.2 Simulations and discussion of results

Using the above two-step FAB schemes (5.8), we present graphical visualizations to demonstrate the behaviour of the approximate solutions to the fractional Zika virus model (3.2) with Caputo derivatives for each system variable. The plots in each of the graphs are with respect to distinct values of the fractional order parameter $\sigma$with σ= 1.0; 0.9; 0.8; 0.7. The time level up to 100 days and the step size used for evaluating the approximate solutions is h = 0.002. The graphs for susceptible individuals $S_h(t)$, exposed individuals $E_h(t)$, infected individuals $I_h(t)$, recovered individuals $R_h(t)$, susceptible vectors $S_v(t)$, exposed vectors $E_v(t)$ and infected vectors $I_v(t)$ are presented in Figure 1-7, respectively. In each of the plots, it is observed that the magnitude of σcontinuously affects the trend of each state variable for both the human and vector populations.

Figure 1- 4: plots shows the dynamics of the susceptible individuals, exposed individual, Infected individual and Recovered individual respectively using the above scheme for the Caputo derivatives. Figure 5-7 demonstrates the dynamics of susceptible vectors, exposed vectors and infected vectors for different values of σ in each of the plots.

Figure 1: As the value of σ increases from 0.7 to 1, there is decrease in the number of susceptible individuals and then a gradual increase after some time until it steadies at equilibrium

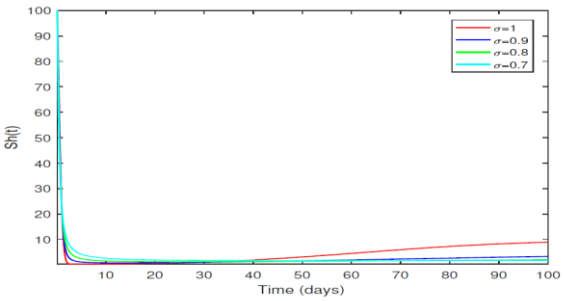


Figure 2 shows the behaviour of the exposed individual using the corresponding $E_h(t)$-schemes of (5.8) for the Caputo derivatives for distinct values of the fractional parameter

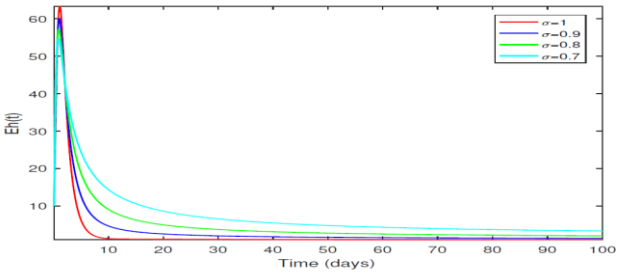


Figure 3: It is observed that as the fractional order parameter increases from 0.7 to 1 the number of infected individual decreases after some time

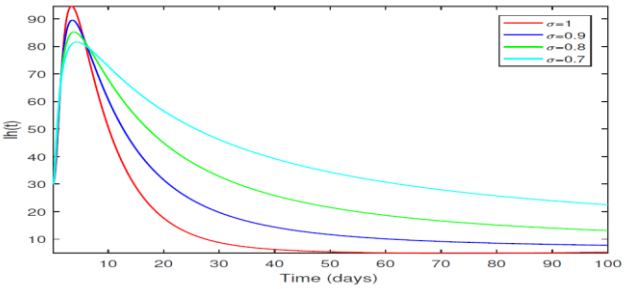
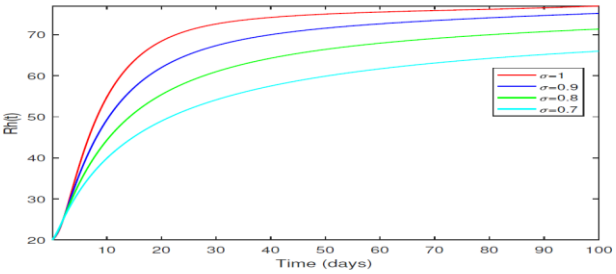


Figure 4 shows the behaviour of the recovered individual using the corresponding $R_h(t)$-schemes of (5.8) for the Caputo derivatives for distinct values of the fractional parameter



Using the corresponding $S_v(t)$-schemes in (5.8) for the Caputo operators, Figure 5 demonstrates the dynamics of susceptible vectors

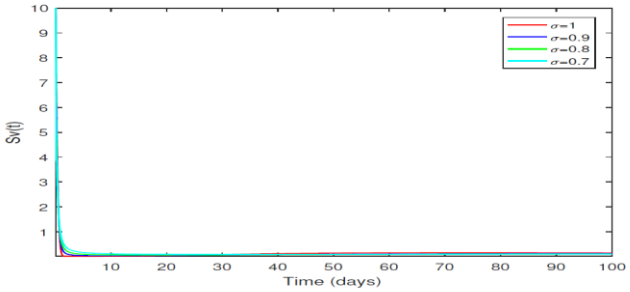
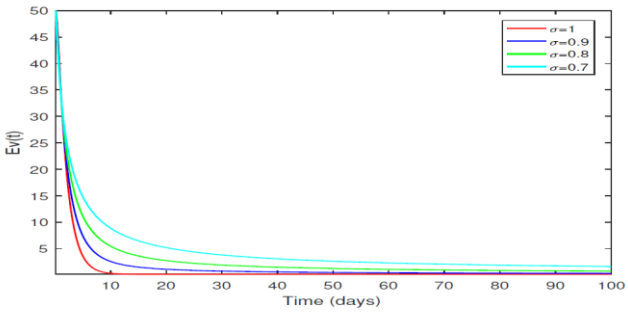
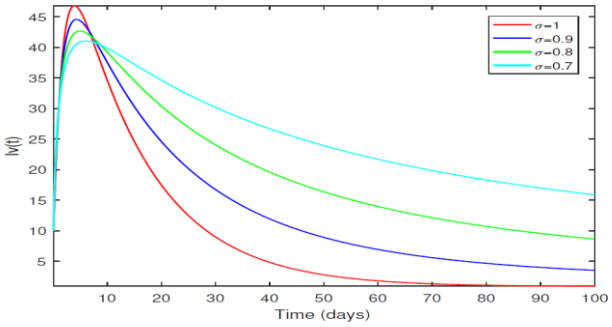


Figure 6: The plots demonstrate the dynamics of the exposed vectors for the case of the $E_v(t)$-equation of (3.2) using the corresponding $E_v(t)$-schemes in (5.8) for the Caputo





we compare the dynamics of Zika infected vectors by presenting plots for the approximate $I_v(t)$-solution using the corresponding $I_v(t)$-schemes of (5.8) for the Caputo for different values of σ. It is observed that as the fractional order parameter increases from 0.7 to 1 the population of infected vectors decreases after some time

6 Conclusions

We analyzed a fractional mathematical model for the transmission dynamics of Zika virus under the framework of singular kernels. The solution set of the classical model is shown to be non-negative and positively invariant. We then determine the equilibrium points of the model and the basic reproduction number is determined via the next generation matrix technique. Existence and uniqueness of solutions to the fractional model with respect to Caputo derivatives are established via a fixed-point technique. Numerical investigations using the two-step Adams-Bashforth method for the fractional Zika virus model with respect to the Caputo fractional differential operators are then carried out with the purpose of demonstrating the dynamics of each of the system variables for different values of the fractional order parameter. We made comparison of the obtained results for each compartment with respect to the Caputo derivative.

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Conflict of interest

The authors declare no potential conflict of interests regarding the publication of this paper.

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