# The Use of Bernstein Polynomial for Solving Fractional Integro-Differential Equations

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**Abstract**

This paper is focused on the study of Bernstein polynomial and the use of it to solve fractional integro differential equations (FIDEs) with caputo derivative. Bernstein polynomial is used to reduce the equation to a system of linear equations from which the approximate solution was obtained. It was observed that the deviation between the approximate solution using Bernstein polynomial and the exact solution is negligible. Graph was presented to show the accuracy of the method.

**Keywords:** Bernstein polynomial, equation, approximate, derivative, fractional.

## Introduction

An integro differential equation is an equation that involves both integrals and derivatives of a function (Losif, 2017). According to Rama (2007), the general first order of linear integro differential equation is of the form;

$$\frac{d}{dx}u(x) + \int_{x_0}^{x} f\big(t,\ u(t)\big)dt = g(x, u(x)), \quad (1)$$

$$\mathrm{u}(x_0) = u_0.$$

According to Awawdeh et al. (2011), fractional calculus was discovered by Leibniz in the year 1695, few years after he discovered classical calculus but later forgotten due to the complexity of the formula. Nanware et al. (2021) recorded that fractional differential equation when compared to integer order differential equation is more advantageous in the sense that it serves as a better model for some natural physical process and system processes because the fractional order differential operators are non-local operators. Nanware et al. (2021) also said that the concept of fractional calculus can be applied in diverse and widespread fields of engineering and sciences such as viscoelasticity, electro-chemistry, fluid mechanics, electro-magnetics and signal processing etc. The role played by fractional integro differential equation cannot be over-emphasized as it models real world problems such as the modelling of earth quakes, reducing the spread of viruses, control the memory behaviour of electric sockets and many others.Since most fractional integro differential equation cannot be solved analytically, much attention has been devoted to search for approximate and numerical techniques for the solution of fractional integro differential equations (oyedepo et.al., 2016). Recently many methods have been develop by researchers for providing approximate solutions of fractional integro differential equations. Osama et al., (2012) employed laguire polynomials as basis function for the solution of fractional solving fredholm integro differential equations. While Aysegul et al.. (2019) employed Bernstein polynomials as basis function to approximate the the solution of fractional integro differential equations. Dilkel et.al., (2018) applied collocation technique for solving fractional integro differential technique using different basis function. Mahdy et.al, (2013) applied sumudu transform method and hermite spectral collocation method for solving fractional integro differential equations. Author Mohammed et.al, (2014) introduced approximate solutions of volterra fredholm integro differential equation of fractional order. Mahdy et.al, (2016) used least square method for the solution of fractional integro differential equation. Mohammed et.al, (2014) introduced numerical solution of fractional singular integro differential equations by using Taylor series expansion and Galerkin method and a fast numerical algorithm based on the second kind of chebychev polynomials. (Seria et.al, 2014) applied numerical solution of fredholm-voltera fractional integro differential equations. Oyedepo et.al, (2021) employed Bernstein modified homotopy perturbation method for the solution of voltera fractional integro differential equation with non local boundary condition. In other to ascertain a better method of solving fractional integro differential equations, the Bernstein polynomial method and homotopy analysis transform method are compared. The homotopy analysis method, introduced first by Liao, is a general approximate analytic approach used to obtain series solutions off nonlinear equations of various types, including algebraic equations, ordinary differential equations, partial differential equations, differential-integral equations,differential-difference equations and and a couple of similar equations. This method is valid no matter whether a non linear problem contains small/large physical parameters or not, which is essentially required in perturbation techniques. More importantly, unlike all perturbation and traditional non-perturbation methods, the homotopy analysis method provides us with both the freedom to choose proper base functions for approximating a nonlinear problem and a sample way to ensure the convergence of the solution series. The use of Berstein polynomials to solve fractional integro-differential equations involves using Bernstein polynomial as basis function to get the approximate solution of the equation.

The general form of the class of problem considered in this work is

$$D^{\propto}u(x) = u(x)p(x) + f(x) + \int_0^x k(x,t)u(x)dt, \quad 0 \le x.t \le 1. \quad (2)$$

For $x \in [0,1]$ with the initial conditions

$$j^{(i)} = \delta_i \qquad i = 0,1,2,\ldots,n-1, \quad n-1 < \propto \leq n, \qquad n \in N.$$

Where $f \in L^2([0,1]),\ u \in L^2([0,1]),\ k \in L^2([0,1]^2)$ are known functions, y(t) is the unknown function.

Where $D^{\propto}$ indicates the $\propto th$ caputo fractional derivative of u(x); p(x); f(x).

K(x,t) are given smooth functions $\delta_j$ are real constant, $x\ and\ t$ are real variables varying [0,1] and u(x) is the unknown function to be determined.

In this work Bernstein polynomial is used to solve fractional integro differential equations by reducing the fractional integro differential equation to a system of linear equations. The approximate solution gotten from this use of Bernstein polynomial is then compare to the exact solution to know how much it deviates from the exact solution. Graphs are used to illustrate the comparison between the approximate solution gotten from the method and the exact solution.

**Definition of basic terms**

**Gamma function**

Gamma function is defined as

r(x)= $\int_0^\infty x^{z-1}e^{-1}\,dx$ (3)

or

r(z+1) = $\int_0^\infty x^z \,.\, e^{-t}dx.$ (4)

This integral converges when the real part of z is positive (Re(z)$\leq$ 0).

Where z is a positive integer

r(z)=(z-1)! (5)

the formula above is use to find the value of the gamma function for any real value of z.

Gamma function according to ustaoglu (2014) has two characteristics first it is definitely an increasing function with respect z and secondly when z is a natural number $\tau(z+1) = z!$

**Beta Function**

A beta function is a kind of function which we classify as the first kind of Euler's integrals. The function has real number domains. We express this function as B(x,y) where x and y are real and greater than 0. The beta function is also symmetric, which means B(x,y)= B(y,x).The notation use for beta function is $\beta$.

Mathematically Beta function is defined as

$$B(x,y)=\int_0^1 (1-u)^{x-1}u^{x-1}du = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = B(x,y), where\ x,y \in R. \qquad (6)$$

**Caputo Fractional Derivative**

The caputo fractional derivative is defined as

$$D^{\propto}(x) = \frac{1}{\Gamma(n-x)} \int_0^x (x-s)^{n-\propto-1} f^m(s) d(s). \tag{7}$$

Where m is a positive integer with the property that

$n-1 < \propto < n.$

For instance, if $0 < \propto < 1$ the caputo fractional derivative is

$$D^{\propto} f(x) = \frac{1}{\Gamma(1-x)} \int_o^x (x-s)^{-\propto} f^1(s) ds.$$

According to oyedepo et al (2016) some of the properties of caputo fractional derivative are,

1. $I^{\propto} I^{v} f = I^{a+v} f, a, v > 0, f \in (Ц, Ц) > 0.$
2. $I^{\propto} x^{y} = \frac{\Gamma(\lambda+1)}{\Gamma(a+y+1)} x^{a+y}, a > 0, y > -1, x > 0.$
3. $I^{\propto} D^{\propto} f(x) = F(x) - \sum_{k=0}^{n-1} f^{k}(0) \frac{x_k}{k!} x > 0, n-1 < \propto \leq n.$
4. $D^{\propto} I^{\propto} f(x) = f(x)\ x > 0, n-1 < a \leq n.$
5. $D^{\propto} C = 0, C\ is\ the\ constant.$

**Bernstein Basis Polynomials**

The n+1 Bernstein basis polynomials of degree n are defined as

$$B_{v,n}(x) = \binom{n}{v} x^v (1-x)^{n-v}, \quad V = 0,1,\dots,n. \tag{8}$$

$$where \binom{n}{v} is\ a\ binomial\ coefficient$$

i.e $\binom{n}{v} = \frac{n!}{v!(n-1)!}$ (9)

for example

$b_{2,5}(x) = \binom{5}{2} x^2 (1-x)^3 = 10x^2(1-x)^3.$

The first few Bernstein basis polynomials for blending 1, 2 ,3 0r 4 values together are

$b_{0,0}(x) = 1,$

$b_{0,1}(x) = 1-x, \qquad b_{1,1}(x) = x,$

$b_{0,2}(x) = (1-x)^2 b_{1,2}(x) = 2x(1-x) b_{2,2}(x) = x^2,$

$b_{0,3}(x) = (1-x)^3, \qquad b_{1,3}(x) = 3x(1-x)^2\ b_{2,3}(x) = 3x^2(1-x).$

The Bernstein basis polynomial of degree n form a basis for the vector space $\Pi_n$

of polynomials of degree of most n with real coefficients.

A linear combination of Bernstein basis polynomials

$B_n(x) = \sum_{v=0}^{n} B_v\, b_v n(x),$
(10)

is called a Bernstein polynomial or polynomial in Bernstein form of degree n. The coefficient $B_v$ is called Bernstein coefficients or Berzier coefficients.

**Absolute Error**

Absolute error in this work is defined as

Absolute error = $|u(x) - u_m(x)|$; $0 \leq x \leq 1$. (11)

Where U(x) is the exact solution and $u_m(x)$ is the approximate solution

And $u_m(x)$ is a Bernstein polynomial of degree m, where $a_j, j = 0, 1, 2, \ldots$

**Illustration of use of Bernstein Polynomials.**

In 2020, Holambe et.al. used Bernstein polynomial as basis function to solve fractional integrate differential equation of the form;

$$D_*^{\propto} u(x) = f(x) + \int_a^b k(x,t)u(t)dt, \quad u(a) = u_a, 0 < a \leq 1. \tag{12}$$

Where $D_*^{\propto} u(x)$indicates the $\propto^{th}$ caputo fractional derivative of $u(x)$. $f(x), k(x,t)$ $are\ given\ functions. x\ and\ t\ are\ r$ . They began the method by taking the fractional integration of both sides of the equation (12) to get;

$$u(x) = u(0) + 1^{\propto} f(x) + 1^{\propto}\left(\int_a^b k(x,t)u(t)dt\right). \tag{13}$$

To determine the approximate solution of (12), they used the Bernstein polynomials basis on $[a, b]$ as

$$u(x) = \sum_{i=0}^{n} a_i B_{i,n}(x). \tag{14}$$

Where $a_i (i = 0,1,2, \ldots, n)$ are unknown constants to be determined.

Substituting equation (14) into equation (13) , they obtained:

$$\sum_{i=0}^{n} a_i B_{i,n}(x) = u(0) + 1^{\propto} f(x) + 1^{\propto}\left(\int_a^b k(x,t) \sum_{i=0}^{n} a_i B_{i,n}(t)dt\right).$$

Hence

$$\sum_{i=o}^{n} a_i B_{i,n}(x) - 1^{\propto}\left(\sum_{i=0}^{n} a_i \varphi(x)\right) = u(0) + 1^{\propto} f(x),$$

Where $\varphi(x) = \int_a^b k(x,t) B_{i,n}(t)dt$,

Substitute the values of $B_{i,n}(x), B_{i,n}(t)$ and simplifyingthe integration.

$$\sum_{i=o}^{n} a_i \left[B_{i,n}(x) - 1^{\propto}\varphi(x)\right] = u(0) + 1^{\propto} f(x), \tag{15}$$

Using the caputo integration and simplifying. Now, we put $x = x_m$, m=0, 1… n into equation (15), $x_m's$ are being chosen as suitable distinct points in (a,b), putting x=xm we obtain the linear system;

$\sum_{i=0}^{n} a_i \, a_{i,j} = \beta j, j = 0,1, \dots, n.$ (16)

Where $a_{i,j} = B_{i,n}(xj)$-$1^{\propto}\varphi(x)$ and Bj= u(0) +$1^{\propto}f(xj)$. Solve the linear system of equations by standard methods for the unknown a's. Substituting ai(i=0,1,…n) in equation (16) to obtain the approximate solution of u(x).

**Example 1;**

The equation to be considered is the fractional integro differential equation

$D^{\propto}y(x) = \frac{8}{3\Gamma(0.5)}x^{1.5} - x^2 - \frac{1}{3}x^3 + \int_0^1 y(t)dt$ . (17)

The exact solution of equation (17) is $x^2$. Using Bernstein polynomial to solve equation (17), we have the following:

$$D^{\propto}y(x) = \frac{8}{3\Gamma(0.5)}x^{1.5} - x^2 - \frac{1}{3}x^3 + \int_0^1 y(t)dt.$$

Taking the fractional integration of both sides of equation (17) we get

$y(x) = I^{\propto}\left(\frac{8}{3\Gamma(0.5)}x^{1.5} - x^2 - \frac{1}{3}x^3\right) + I^{\propto}\left(\int_0^1 y(t)\right)dt$ . (18)

To determine the approximate solution of equation (18) we get

$\sum_{i=0}^{3} a_i b_{i,3}(x) = I^{\propto}\left(\frac{8}{3\Gamma(0.5)}x^{1.5} - x^2 - \frac{1}{3}x^3\right) + I^{\propto}\left(\int_0^1 \sum_{i=0}^{3} a_i b_{i,3}(t)\right)dt,$
(19)

so,

$a_0(1-x)^3 + a_1 3x(1-x)^2 + a_2 3x^2(1-x) + a_3 x^3 - I^{\propto}\left[a_0\int_0^1(1-t)^3 dt + 3a_1 t\int_0^1(1-t)^2 dt + 3a_3 t^2\int_0^1(1-t)dt + a_3\int_0^1 t^3 dt\right] = I^{\propto}\left(\frac{8}{3\Gamma(0.5)}x^{1.5} - x^2 - \frac{1}{3}x^3\right),$ (20)

evaluating the right hand side of equation (20), we have

$a_0(1-x)^3 + a_1 3x(1-x)^2 + a_2 3x^2(1-x) + a_3 x^3 - I^{\propto}\left[a_0\int_0^1(1-x)^3\,dt + 3a_1\int_0^1(t^2 + t^3 - 2t^2)dt + 3a_2\int_0^1(t^2 - t^3)dt + a_3\int_0^1 t^3 dt\right] = \frac{8}{3\Gamma(0,5)} \times \frac{\Gamma(2.5)}{\Gamma(2.5+\propto)}x^{\propto+1.5} - \frac{\Gamma(3)}{\Gamma(3+\propto)}x^{\propto+2} - \frac{1}{3} \times \frac{\Gamma(4)}{\Gamma(4+\propto)}x^{\propto+3}.$ (21)

Simplifying the RHS further we have

$a_0(1-x)^3 + a_1 3x(1-x)^2 + a_2 3x^2(1-x) + a_3 x^3 - I^{\propto}\left[a_0\int_0^1(1-x)^3\,dt + 3a_1\int_0^1(t^2 + t^3 - 2t^2)dt + 3a_2\int_0^1(t^2 - t^3)dt + a_3\int_0^1 t^3 dt\right] = 2\frac{x^{\propto+1.5}}{\Gamma(2.5+\propto)} - 2\frac{x^{\propto+2}}{\Gamma(3+\propto)} - 2\frac{x^{\propto+3}}{\Gamma(4+\propto)}$(22)

$a_0(1-x)^3 + a_1 3x(1-x)^2 + a_2 3x^2(1-x) + a_3 x^3 - I^{\propto} a_0\{(1-t)^3 + 3(1-t)^2 + 6(1 = t) + 6\}_0^1 + 3a_1\{(t + t^2 - 2t^3) + (1 + 2t^2 - 4t) + (6t - 4) + 6\}_0^1 + 3a_2\{(t^2 - t^3) + (2t - 3t^2) + (t - 6t) - 6\}_0^1 + a_3\{t^3 + 3t^2 + 6t - 6\}_0^1 = 2\frac{x^{\propto+1.5}}{\Gamma(2.5+\propto)} - 2\frac{x^{\propto+2}}{\Gamma(3+\propto)} - 2\frac{x^{\propto+3}}{\Gamma(4+\propto)}$, (23)

$a_0(1-x)^3 + a_1 3x(1-x)^2 + a_2 3x^2(1-x) + a_3 x^3 - [a_0(-10]\frac{x^{\propto}}{\Gamma(1+\propto)} + a_1(-3)\frac{x^{\propto}}{\Gamma(1+\propto)} + a_2(-18)\frac{x^{\propto}}{\Gamma(1+\propto)} + a_3(3)\frac{x^{\propto}}{\Gamma(1+\propto)} = 2\frac{x^{\propto+1.5}}{\Gamma(2.5+\propto)} - 2\frac{x^{\propto+2}}{\Gamma(3+\propto)} - 2\frac{x^{\propto+3}}{\Gamma(4+\propto)}$, (24)

$a_0\left[(1-x)^3 + 10\frac{x^{\propto}}{\Gamma(1+\propto)}\right] + a_1\left[3x(1-x)^2 + 3\frac{x^{\propto}}{\Gamma(1+\propto)}\right] + a_2\left[3x^2(1-x) + 18\frac{x^{\propto}}{\Gamma(1+\propto)}\right] + a_3\left[x^3 - 3\frac{x^{\propto}}{\Gamma(1+\propto)}\right] = 2\frac{x^{\propto+1.5}}{\Gamma(2.5+\propto)} - 2\frac{x^{\propto+2}}{\Gamma(3+\propto)} - 2\frac{x^{\propto+3}}{\Gamma(4+\propto)}$, (25)

substitute $\propto = 0.5$ into equation (25)

$a_0\left[(1-x)^3 + 10\frac{x^{0.5}}{\Gamma\left(\frac{3}{2}\right)}\right] + a_1\left[3x(1-x)^2 + 3\frac{x^{0.5}}{\Gamma\left(\frac{3}{2}\right)}\right] + a_2\left[3x^2(1-x) + 18\frac{x^{0.5}}{\Gamma\left(\frac{3}{2}\right)}\right] + a_3\left[x^3 - 3\frac{x^{0.5}}{\Gamma\left(\frac{3}{2}\right)}\right] = 2\frac{x^{25}}{\Gamma(23)} - 2\frac{x^{2.52}}{\Gamma\left(\frac{7}{2}\right)} - 2\frac{x^{3.5}}{\Gamma\left(\frac{9}{2}\right)}$.

Then substituting x = 0.1, 0.2, 0.3 and 0.4 respectively, we get a linear system of equations,

$$4.287a_0 + 1.3134a_1 + 6.4494a_2 - 1.0694a_3 = 0.0080$$

$$5.558a_0 + 1.8978a_1 + 9.1788a_2 - 1.5058a_3 = 0.0286$$

$$6{,}552a_0 + 2{,}2947a_1 + 11.3112a_2 - 1.8267a_3 = 0.0576$$

$$7.352a_0 + 2.5728a_1 + 13.1328a_2 - 2.0768a_3 = 0.092$$

Therefore, using maple 20 to solve the above system of equation we get

$$a_0 = -0.00918053195$$

$$a_1 = 0.02356660755$$

$$a_3 = 0.12100236583$$

$$a_3 = 0.71440817286$$

Thus, the approximate solution of equation (17) when $\propto = 0.5$ becomes

$$y(x) = -0.00918(1-x)^3 + 0.0236(3x)(1-x)^2 + 0.1210(3x^2)(1-x) + 0.7144(x^3)$$

Table 1 showing the exact and approximate solution of equation (17) using Bernstein polynomial method

| X | Exact solution | Approximate solution of Bernstein polynomials method |
|---|---|---|
| 0.1 | 0.01 | 0.0030167 |
| 0.2 | 0.04 | 0.01938056 |
| 0.3 | 0.09 | 0.0494037 |
| 0.4 | 0.16 | 0.0887696 |
| 0.5 | 0.25 | 0.1423675 |
| 0.6 | 0.36 | 0.2127852 |
| 0.7 | 0.49 | 0.3026105 |
| 0.8 | 0.64 | 0.4144312 |
| 0.9 | 0.81 | 0.5508351 |

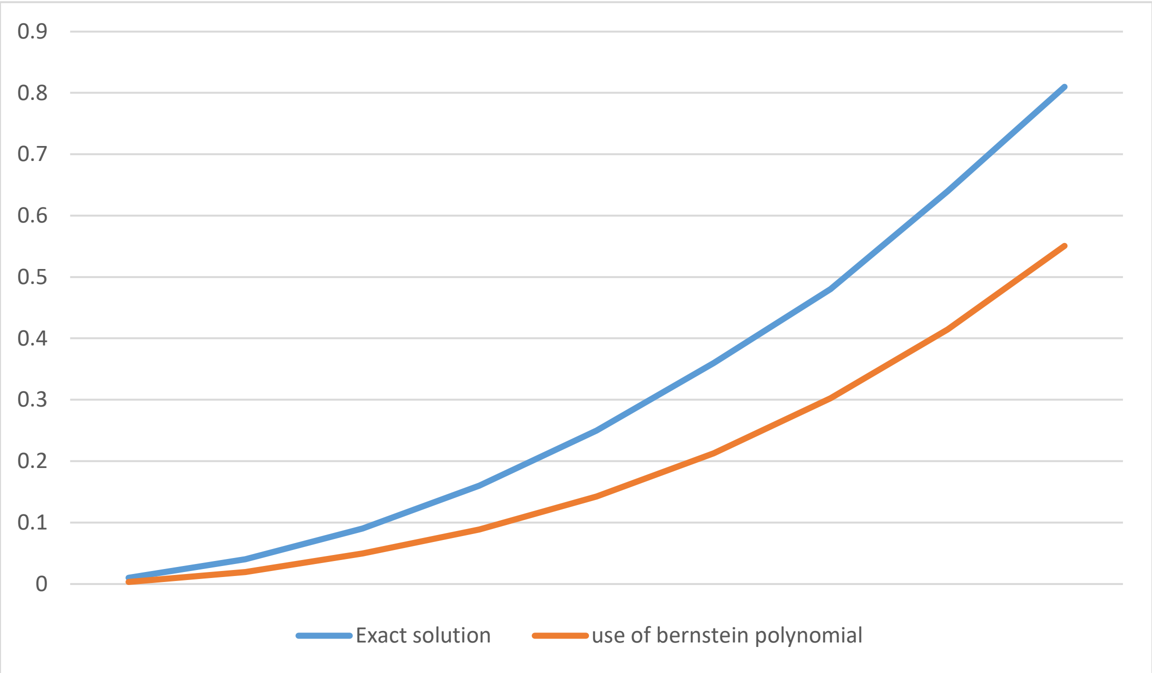


Figure 1 showing the graph of approximate solution from the use of Bernstein polynomial and exact solution.

Table 2 showing the absolute error from the use of Bernsteim polynomial

| X | EXACT SOLUTION | APPROXIMATE SOLUTION FROM THE USE OF BERNSYEIN POLYNOMIAL | ABSOLUTE ERROR FROM THE USE OF BERNSTEIN POLYNOMIAL |
|---|---|---|---|
| 0.1 | 0.01 | 0.0030167 | 0.0069833 |
| 0.2 | 0.04 | 0.01938056 | 0.02061944 |
| 0.3 | 0.09 | 0.0494037 | 0.0403963 |
| 0.4 | 0.16 | 0.0887696 | 0.0712304 |
| 0.5 | 0.25 | 0.1423675 | 0.1076325 |
| 0.6 | 0.36 | 0.2127852 | 0.1472148 |
| 0.7 | 0.49 | 0.3026105 | 0.1873895 |
| 0.8 | 0.64 | 0.4144312 | 0.2255688 |
| 0.9 | 0.81 | 0.5508351 | 0.2591649 |

**Example 2;**

Consider the fractional integro-differential equation

$$D_*^\alpha y(x) = sinx + e^{3x} + \int_0^1 xe^t y(t)dt \quad (26)$$

Taking the fractional integration of both sides of the equation, we get

$$y(x) = y(0) + I^\alpha(sinx + e^{3x}) + I^\alpha(\int_0^1 xe^t y(t))dt \quad (27)$$

To determine the approximate solution of (26), we say

$$y(x) = \sum_{i=0}^{3} a_i B_{i,3}(x)$$

And after substituting into equation (27), we get

$$\sum_{i=0}^{3} a_1 B_{i,3}(x) = I^\alpha(\sin x + e^{3x}) + I^\alpha\left(\int_0^1 xe^t \sum_{i=0}^{3} a_i B_{i,3}(t)dt\right)$$

So,

$a_0(1-x)^3 + a_1 3x(1-x)^2 + a_2 3x^2(1-x) + a_3 x^3 - I^\alpha \Big[a_0 x \int_0^1 (1-t)^3 e^t\, dt +$
$3a_1 x \int_0^1 (t + t^3 - 2t^2) e^t\, dt + 3a_2 x \int_0^1 (t^2 - t^3) e^t\, dt - a_3 x \int_0^1 t^3\, e^t dt\Big] = I^\alpha(sinx + e^{3x})$

Using the maclaurin series up to five terms on the right hand side, we get

$a_0(1-x)^3 + a_1 3x(1-x)^2 + a_2 3x^2(1-x) + a_3 x^3 - I^\alpha \Big[a_0 x \int_0^1 (1-t)^3 e^t\, dt +$
$3a_1 x \int_0^1 (t + t^3 - 2t^2) e^t\, dt + 3a_2 x \int_0^1 (t^2 - t^3) e^t\, dt - a_3 x \int_0^1 t^3\, e^t dt\Big] = I^\alpha \Big(1 + 4x +$
$\frac{9}{2}x^2 + \frac{13}{3}x^3 + \frac{27}{8}x^4 + \frac{81}{40}x^5\Big)$

$a_0(1-x)^3 + a_1 3x(1-x)^2 + a_2 3x^2(1-x) + a_3 x^3 - I^\alpha a_0 x\{(1-t)^3(e^t) - 3(1-$
$t)^2(-1)e^t + 6(1-t)(e^t) + 6e^t\}\ {}_0^1 + 3a_i x\{(t + t^3 - 2t^2)e^t - (1 - 3t^2 - 4t)e^t + (6t -$
$4)e^t - (6)e^t\}_0^1 + 3a_2 x\{(t^2 - t^3)e^t - (2t - 3t^2)e^t + (2 - 6t)e^t + 6e^t\}_0^1 = I^\alpha \Big(1 + 4x +$
$\frac{9}{2}x^2 + \frac{13}{3}x^3 + \frac{27}{8}x^4 + \frac{81}{40}x^5\Big)$

$a_0(1-x)^3 + a_1 3x(1-x)^2 + a_2 3x^2(1-x) + a_3 x^3 - a_0(6e - 16)\frac{\Gamma(2)}{\Gamma(2+\propto)}x^{1+\propto} -$
$3a_1(11 - 4e)\frac{\Gamma(2)}{\Gamma(2+\alpha)}x^{1+\alpha} - a_3(6 = 6e)\frac{\Gamma(2)}{\Gamma(2+\alpha)}x^{1+\alpha} = \frac{x^\alpha}{\Gamma(\alpha+1)} + 4\frac{x^{1+\alpha}}{\Gamma(\alpha+4)} + 9\frac{x^{2+\alpha}}{\Gamma(\alpha+3)} +$
$26\frac{x^{3+\alpha}}{\Gamma(\alpha+4)} + 81\frac{x^{4+\alpha}}{\Gamma(\alpha+5)} + 243\frac{x^{5+\alpha}}{\Gamma(\alpha+6)}$

$a_0\Big[(1-x)^3 - (6e - 16)\frac{x^{1+\alpha}}{\Gamma(2+\alpha)}\Big] + a_1\Big[3x(1-x)^2 - (11 - 4e)\frac{x^{1+\alpha}}{\Gamma(2+\alpha)}\Big] + a_2\Big[3x^2(1-$
$x) - 3(3e - 8)\frac{x^{1+\alpha}}{\Gamma(2+\alpha)}\Big] + a_3\Big[x^3 - (6 - 2e)\frac{x^{1+\alpha}}{\Gamma(2+\alpha)}\Big] = \frac{x^\alpha}{\Gamma(\alpha+1)} + 4\frac{x^{1+\alpha}}{\Gamma(\alpha+4)} + 9\frac{x^{2+\alpha}}{\Gamma(\alpha+3)} +$
$26\frac{x^{3+\alpha}}{\Gamma(\alpha+4)} + 81\frac{x^{4+\alpha}}{\Gamma(\alpha+5)} + 243\frac{x^{5+\alpha}}{\Gamma(\alpha+6)}$

$a_0\Big[(1-x)^3 - (6e - 16)\frac{x^{1+\alpha}}{\Gamma(2+\alpha)}\Big] + a_1\Big[3x(1-x)^2 - (11 - 4e)\frac{x^{1+\alpha}}{\Gamma(2+\alpha)}\Big] + a_2\Big[3x^2(1-$
$x) - 3(3e - 8)\frac{x^{1+\alpha}}{\Gamma(2+\alpha)}\Big] + a_3\Big[x^3 - (6 - 2e)\frac{x^{1+\alpha}}{\Gamma(2+\alpha)}\Big] = \frac{x^{0.5}}{\Gamma(1.5)} + 4\frac{x^{1.5}}{\Gamma(2.5)} + 9\frac{x^{2.5}}{\Gamma(3.5)} + 26\frac{x^{3.5}}{\Gamma(4.5)} +$
$81\frac{x^{4.5}}{\Gamma(5.5)} + 243\frac{x^{5.5}}{\Gamma(6.6)}$

(28)

Substituting x=0.1, 0.2, 0.3,and 0.4 into equation (28) respectively, we get a linear system that has the following;

0.721633$a_0$+0.233947$a_1$+0.0159495$a_2$-0.0124032$a_3$= 0.4613003634

0.4911629$a_0$+0.3583907$a_1$+0.0647443$a_2$-0.02991$a_3$= 0.83143000363

0.3047198$a_0$+0.39527$a_1$+0.0.1315797$a_2$-0.0426451$a_3$= 1.28450996363

0.41570638$a_0$+0.3595659$a_1$+0.1995957$a_2$-0.0432256$a_3$= 1.86991639505

Solving the system of equations above with MAPLE 18 gives

$a_0$= 0.02422727

$a_1$= 2.5206785

$a_2$= 10.194955798

$a_3$= 24.8720339006

Thus the approximate solution of equation (26) when $\alpha = 0.5$ is

$y(x) = 0.02422726997(1-x)^3 + 2.5206784693(3x)(1-x)^2 + 10.19495579849(3x^2)(1-x) + 24.87203390075(x^3)$

Table 3 showing the exact and approximate solution of equation (26) with the use of Bernstein polynomial

| X | Exact solution | Approximate solution from the use of Bernstein polynomial |
|---|---|---|
| 0.1 | 0.3 | 0.92807099176 |
| 0.2 | 1.0 | 2.1588049496 |
| 0.3 | 2.1 | 3.7192027439 |
| 0.4 | 3.6 | 5.6229876488 |
| 0.5 | 5.5 | 7.881145475 |
| 0.6 | 7.8 | 10.5046620416 |
| 0.7 | 10.5 | 13.504523465 |
| 0.8 | 13.6 | 16.8917153 |
| 0.9 | 17.1 | 20.677223495 |

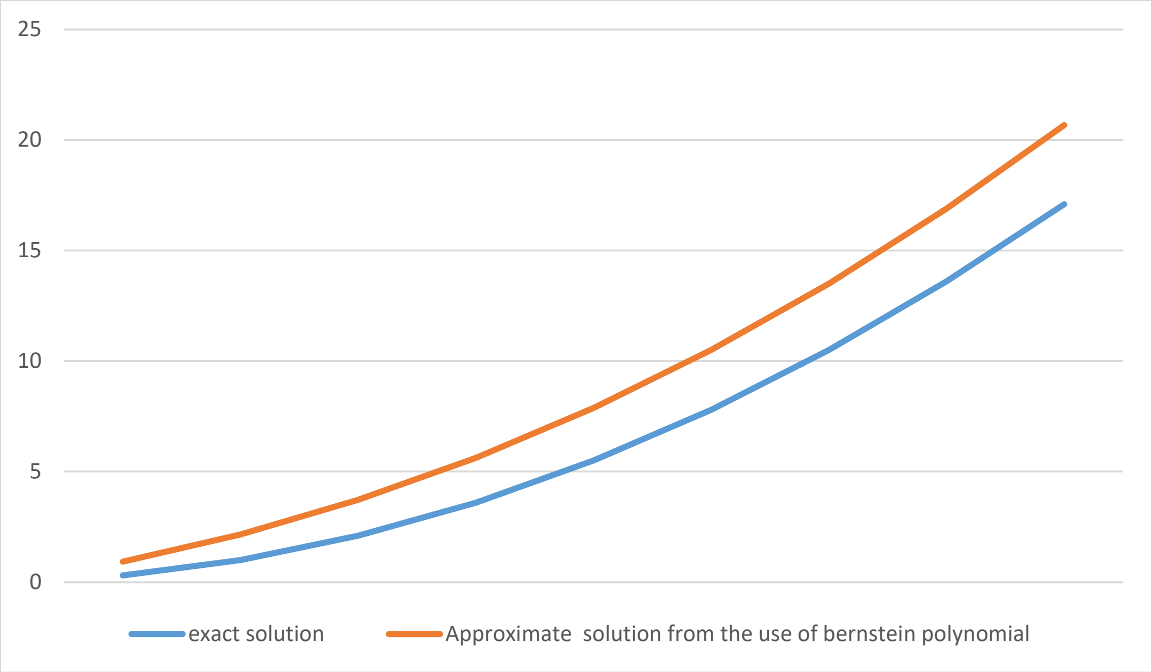


Figure 2 showing the graph of approximate solution from Bernstein polynomial and the exact solution.

Table 4 showing the absolute error from the use of Bernstein polynomial

| X | EXACT SOLUTION | APPROXIMATE SOLUTION FROM THE USE OF BERNSYEIN POLYNOMIAL | ABSOLUTE ERROR FROM THE USE OF BERNSTEIN POLYNOMIAL |
|---|---|---|---|
| 0.1 | 0.3 | 0.92807099176 | -0.62807099176 |
| 0.2 | 1.0 | 2.1588049496 | -i.1588049496 |
| 0.3 | 2.1 | 3.7192027439 | -1.61920274439 |
| 0.4 | 3.6 | 5.6229876488 | -2.0229876488 |
| 0.5 | 5.5 | 7.881145475 | -2.381145475 |
| 0.6 | 7.8 | 10.5046620416 | -3.004523465 |
| 0.7 | 10.5 | 13.504523465 | -3.2917153 |
| 0.8 | 13.6 | 16.8917153 | -3.577223495 |
| 0.9 | 17.1 | 20.677223495 | 0.2591649 |

**Discussion**

This work is focused on the use of Bernstein polynomial for solving fractional integro differential equation at$\propto= 0.5$, the systems of equation gotten in example 1 and 2 were solved using MAPLE 18. Table 1 shows the approximate solution of equation (17) at x=0.1 down to 0.9 and when compared to the exact solution it is observed that the exact solution increases as the value of x increase and so does the approximate solution increase as the value of x increases as well. The difference which is the absolute error as shown in table 2 is not much. In figure 1 a graphical representation of the solution gotten from the use of Bernstein polynomial and exact solution is shown and from the graph one can see that the solution are close at lower values of xand grow bigger as the value of x increases. Table 2 also show the solution of equation (26) from the use of Bernstein polynomial and the exact solution. This time around the exact solution have smaller figures at different value of x compared to the approximate solution gotten from the use of Bernstein polynomial unlike in table 1 of example 1which shows the exact solution having bigger figures at different values of x when compared to the approximte solution. Table 3 also show that as the values of the exact solution increases the values of the approximate solution increases as well. Table 4 shows the absolute error of the solutions and it is observed that the difference between the exact solution and the approximate solution is not much.. figure 2 shows the graphical representation of the exact and approximate solution and like in figure 1 it is observed that at lower values of x the solution are close but grow bigger as the value of x increases.

**Conclusion**

The study applied the use of Bernstain polynomial to find the solution of fractional integro differential equations. The method was use to solve two different problems. The result obtained was compared to the exact solution of and the deviation negligible. Also the result was presented in graphical form to further illustrate the accuracy of the method.

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