Modified variational iteration method for the solution of the time-fractional Nagumo equation

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Accepted 12th December, 2021

To find the numerical solution of the Time-Fractional Nagumo equation, this paper modifies the He's Variational Iteration Method (VIM) using the newly developed Mamadu-Njoseh polynomials (MNPs) as basis function and modifier. We first decompose the nonlinear part using the correction functional of the VIM, and hence treat both as a single linear equation. The fractional part is handled via the Riemann-Liouville sense. It was discovered that the modified VIM (MVIM) was able to approximate the Time-fractional Nagumo equation to its exact solution as the results of this study were compared with those found in literature.

Key words: Fractional differential equation, Time-Fractional Nagumo equation, Variation Iteration Method (VIM), Mamadu-Njoseh Polynomials (MNPs), Orthogonal Collocation Method (OCM)

INTRODUCTION

G.W. Leibniz (1697) used the notation $d^{\frac{1}{2}}y$ to represent the derivative of order $\frac{1}{2}$ as against the classical calculus of having an integer order derivative. However, N.H. Abel was the first to introduce the form $\int_0^x (x-t)^{-\frac{1}{2}} f(t)dt$ into fractional calculus in 1823 (Oldham. and Spanier, 1974). Since then, fractional calculus has been applied to majorly modelling of physical phenomenon and for the solutions of real-life problems especially in engineering and the physical sciences. A construction of a fractional model for the treatment of cancer via the Hadamand fractional derivative was also done (Awadalla et al., 2019). The fractional sub-equation method for arriving at the exact analytical solutions for the space-time fractional Benjamin-Bona-Mahony (BBM) equation and the space-time fractional Zakharov-Kuznestov-Benjamin-Bona-Mahony (ZKBBM) equation were constructed in Alzaidy (2013), while Li et al. (2012) investigated on the Cauchy problem for class of time-fractional differential equation where they concluded by providing the exact solution for a kind of generalized fractional telegraph equation. A differential equation for the diffusion in isotopic and homogeneous fractal structures was derived within the context of fractional calculus (Giona and Roman, 1997), while Hilfer (2000) developed the Adomian decomposition Natural Transform Method (ADTNM) which is a combination of the Adomian Decomposition Method (ADM) and the Natural Transform Method (ATM) to obtain an approximate analytic solution of fractional physical models using the Caputo fraction derivatives. The Variational Iteration Method (VIM) developed by He (1999) is an iterative method for solving linear and nonlinear equations whereby the nonlinear term is treated like the linear term without any unrealistic assumption. He et al. (2010) state that the VIM's edge over other iterative methods is to construct a correction functional for the nonlinear system given as

$$y_{n+1} = y_n(x,t) + \int_0^t \lambda\left(Ly_n(\tau) + N\tilde{y}_n(\tau) - g(\tau)\right) d\tau \qquad (1.1)$$

where λ is the Lagrange multiplier to be identified optimally by integration by parts (Wazwaz, 2007). y_n is the nth approximation solution while $\tilde{y}_n$ is the restricted variation, hence $\delta\tilde{y}_n = 0$. The VIM was modified and applied to find the solution of some nonlinear, nonhomogeneous differential equations (Abassy, 2012), while Biazar et al. (2015) made a comparison between the VIM, ADM and HPM and concluded that the VIM is more, convenient, stable and efficient than the other two methods. Its modified form was used to obtain the solution of a fourth order parabolic partial differential equation in Elsheikh and Elzaki (2016) and Onyeoghane and Njoseh (2020) obtained the solutions of Heston stochastic partial differential equation via the MVIM. The MNPs is an orthogonal polynomial developed by Mamadu and Njoseh (2016) to determine the numerical solutions of Volterra equations using Galerkin method, and also finding the numerical solutions of fifth order Boundary Value problems (Njoseh and Mamadu (2016). Onyeoghane and Njoseh (2019) applied the MNPs as a modifier to the Homotopy Perturbation Method and Adomian Decomposition Method to find the approximate solution of the Heston Stochastic Partial Differential Equation.

Here, we modify the VIM using the MNPs constructed within the interval of $[-1,1]$ with a weight function $w(x) = 1 + x^2$ to derive the approximate solution of the time fractional Nagumo equation characterised by linear and nonlinear expressions.

FRACTIONAL DIFFERENTIAL EQUATION

Fractional differential equation or more generally known as fractional calculus is that branch of mathematics that focuses on derivatives and integrals of arbitrary order. From our classical calculus, we have that

$$y = f(x) = \frac{d^n y}{dx^n} \quad , n > 0 \qquad (2.1)$$

In fractional calculus however, we have that,

$$y = f(x) = \frac{d^\alpha y}{dx^\alpha} \qquad (2.2)$$

where α is an arbitrary quantity which may represent real non-negative integer, positive fraction, negative integer, negative fraction or complex numbers (Mathai and Haubold, 2017). Fractional calculus is defined as the "theory and applications of differentiation and integration to arbitrary order" (Oldham and Spanier, 1974).

The Riemann-Liouville fractional integral of order $\alpha > 0$ is given as

$$J^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}\, d\tau, \quad t > 0, \alpha \in \mathbb{R}^+ \qquad (2.3)$$

where $\Gamma(\alpha)$ represents the Gamma function, and α is any arbitrary positive real number.

Also, the Riemann-Liouville fractional derivative of order $\alpha > 0$ is given as

$$D^\alpha f(t) = D^m J^{m-\alpha} f(t) \qquad (2.4)$$

where

$$D^m J^{m-\alpha} f(t) = \begin{cases} \frac{d^m}{dt^m}\left[\frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}}\, d\tau\right] & , m-1 < \alpha < m \\ \frac{d^m}{dt^m} f(\tau), & \alpha = m \end{cases} \qquad (2.5)$$

and

$$D^0 = J^0 = I \qquad (2.6)$$

hence,

$$D^\alpha J^\alpha = I, \quad \alpha \geq 0 \qquad (2.7)$$

and guided by the following principles (Gorenflo and Mainardi, 2000)

$$D^\alpha J^\eta = \frac{\Gamma(\eta+1)}{\Gamma(\eta+1+\alpha)} t^{\eta+\alpha}, \alpha > 0, \eta > -1, t > 0$$

$$J^\eta J^\beta f(x) = J^{\eta+\beta} f(x)$$
$$J^\eta J^\beta f(x) = J^\beta J^\eta f(x);\ \eta, \beta \geq 0 \qquad (2.8)$$

NAGUMO EQUATION

The general Nagumo equation is characterised in finding the solutions of a wavelike equation, as in the shape and speed of pulses in the nerve of a squid, equations of travelling waves and solutions of the space-clamped case (McKean Jr, 1969). Thus, the general Nagumo equation is given as

$$\frac{\partial y_n}{\partial t} = \frac{\partial^2 y_n}{\partial x^2} + f - b \int y_n\, dt \qquad (3.1)$$

where f is a cubic polynomial such that

$$f(y_n) = y_n(1-y_n)(y_n - k), (0 < k, < 1), \qquad (3.2)$$

b is a positive constant and $\int y_n\, dt$ is the indefinite integral.

Simplifying (3.1), gives the Nagumo equation (Srivastava et al., 2018) as

$$y_n(t) = y_n'' + y_n(1-y_n)(y_n - k) \qquad (3.3)$$

involving the time variable, t.

Obtaining the time-fractional Nagumo equation, we set $y_n(t)$ in (3.3) to $D^\alpha y_n(t)$ where $n-1<\alpha\le n\ (n\in\mathbb{N})$, such that the Time-Fractional Nagumo equation is given as

$$D^\alpha y_n(t)=y_n''+y_n(1-y_n)(y_n-k) \qquad (3.4)$$

VARIATIONAL ITERATION METHOD (VIM)

Given the nth order differential equation,

$$\frac{d^n y}{dx^n}+\alpha\frac{d^{n^*}y}{dx^{n^*}}=g(x,t) \qquad (4.1)$$

we set $L=\frac{d^n}{dx^n}$ and $N=\alpha\frac{d^{n^*}}{dx^{n^*}}$, then we have (4.1) as

$$Ly(x,t)+Ny(x,t)=g(x,t) \qquad (4.2)$$

where L is a linear operator, N is a nonlinear operator and g is the source term.

As given in (1.1), we now have the correction functional given as

$$y_{n+1}=y_n(x,t)+\int_0^t\lambda\left(Ly_n(\tau)+N\tilde{y}_n(\tau)-g(\tau)\right)d\tau \qquad (4.3)$$

The Lagrange multiplier can be used to construct a correction functional which can be optimally identified by variational theory and easily determined by the restricted variations in the correction functional. With an unknown constant, the initial approximation can be freely selected, and the constants can be determined through various methods such that the validity of the approximation obtained by the VIM is sure for both small and large parameters with the first-order approximation being of extreme accuracy. Furthermore, the method converges faster in comparison with that of the Adomian Decomposition Method (ADM) (He, 1999).

MAMADU-NJOSEH POLYNOMIALS (MNPs)

Constructed within the interval $[-1,1]$ with a weight function $w(x)=1+x^2$, the MNPs orthogonal relation is given as

$$\int_{-1}^{1}\varphi_m(x)\varphi_n(x)(1+x^2)\,dx=0 \qquad (5.1)$$

based on the following properties

i. $\varphi_n(x)=\sum_{r=0}^{n}C_r^{(n)}x^r$

ii. $\langle\varphi_m(x),\varphi_n(x)\rangle=0,\ m\neq n$

iii. $\varphi_n(x)=1$ (5.2)

where $\varphi_n(x), n\ge 0$ are the orthogonal polynomials.

The first MNPs are given as

$$\varphi_0(x)=1$$
$$\varphi_1(x)=x$$
$$\varphi_2(x)=\frac{1}{3}(5x^2-2)$$
$$\varphi_3(x)=\frac{1}{5}(14x^3-9x)$$
$$\varphi_4(x)=\frac{1}{648}(333-2898x^2+3213x^4)$$
(5.3)

The MNPs will be serving as a modifier for the VIM in finding the approximate solution of the Time-Fractional Nagumo Equation. A lemma was therefore established to guarantee the existence of the MVIM using the MNPs as modifier.

We define the Banach space as $(B(S),\|.\|)$, which is the space of all continuous functions $y_n(x,t)$ with the norm given as

$$\|y_n(x,t)\|=\max_{(x,t)\in S}|y_n(x,t)|, \qquad (5.4)$$

Lemma: Let $y_n(x,t)$ and $y_n^\zeta(x,t);\zeta=1,2,\dots$ be continuous. Then the derivative $D_t^\alpha y_n(x,t)$ is bounded.

Proof: We show that $D_t^\alpha y_n(x,t)$ is bounded. Recalling the Riemann-Liouville fractional derivative from (2.4) and (2.5), we have that

$$\|D^m J^{m-\alpha}f(t)\|=\left\|\frac{d^m}{dt^m}\left[\frac{1}{\Gamma(m-\alpha)}\int_0^t\frac{f(\tau)}{(t-\tau)^{\alpha+1-m}}d\tau\right]\right\| \qquad (5.5)$$

$$\le\frac{|t-t_0|}{|(m-\alpha)\Gamma(m-\alpha)|}\|y_n(x,t)\|$$

$$=L_1\|y_n(x,t)\| \qquad (5.6)$$

such that

$$L_1=\frac{|t-t_0|}{|(m-\alpha)\Gamma(m-\alpha)|} \qquad (5.7)$$

Hence, the proof is complete.

MVIM FOR THE TIME-FRACTIONAL NAGUMO EQUATION

The Nagumo equation (5.2) is subject to the initial condition

$$y_n(x,0) = \frac{1}{2}\left[1 + k - (k-1)tanh\left(\frac{(k-1)x}{2\sqrt{2}}\right)\right] \quad (6.1)$$

We assume a trial solution

$$y_n(x) = \sum_{i=0}^{n} a_i\varphi_i(x) = \frac{1}{2}\left[1 + k - (k-1)tanh\left(\frac{(k-1)x}{2\sqrt{2}}\right)\right], n = 3 \quad (6.2)$$

hence,

$$y_n(x) = \sum_{i=0}^{3} a_i\varphi_i(x) = \frac{1}{2}\left[1 + k - (k-1)tanh\left(\frac{(k-1)x}{2\sqrt{2}}\right)\right] \quad (6.3)$$

Employing the Orthogonal Collocation Method (OCM) and collocating (6.3) at $\varphi_4(x) = 0$, we have that

$x = 0.3676425560, -0.3676425560, 0.8756710201, -0.8756710201$

and writing the resulting linear algebraic equations in the form

$$A\underline{X} = \underline{b} \quad (6.4)$$

where ,

$$A = \begin{bmatrix} 1 & 0.3676425560 & -0.4413982517 & -0.5226219309 \\ 1 & -0.3676425560 & -0.4413982517 & 0.5226219309 \\ 1 & 0.8756710201 & 0.6113328923 & 0.303892222 \\ 1 & -0.8756710201 & 0.6113328923 & -0.303892222 \end{bmatrix}$$

$$\underline{X} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$\underline{b} = \begin{bmatrix} 52.98313121 \\ 52.98313121 \\ 300.5854963 \\ 300.5854963 \end{bmatrix}$$

we solve the equations via the Gaussian Elimination method with the $a_i's, i = 0, \ldots, 3$.
Substituting the $a_i's$ in (6.3) gives us our initial approximation $y_0(x)$
We give the variational scheme for the time fractional Nagumo equation as

$$y_{n+1} = y_n(x,t) + \int_0^t \lambda(x,t)(D^\alpha y_n(t) - y_n'' + y_n(1-y_n)(y_n - k))\,dt \quad (6.5)$$

Setting the Lagrange multiplier $\lambda(x,t) = -\frac{1}{\Gamma(\alpha)}(t-\tau)^{\alpha-1}$, we have

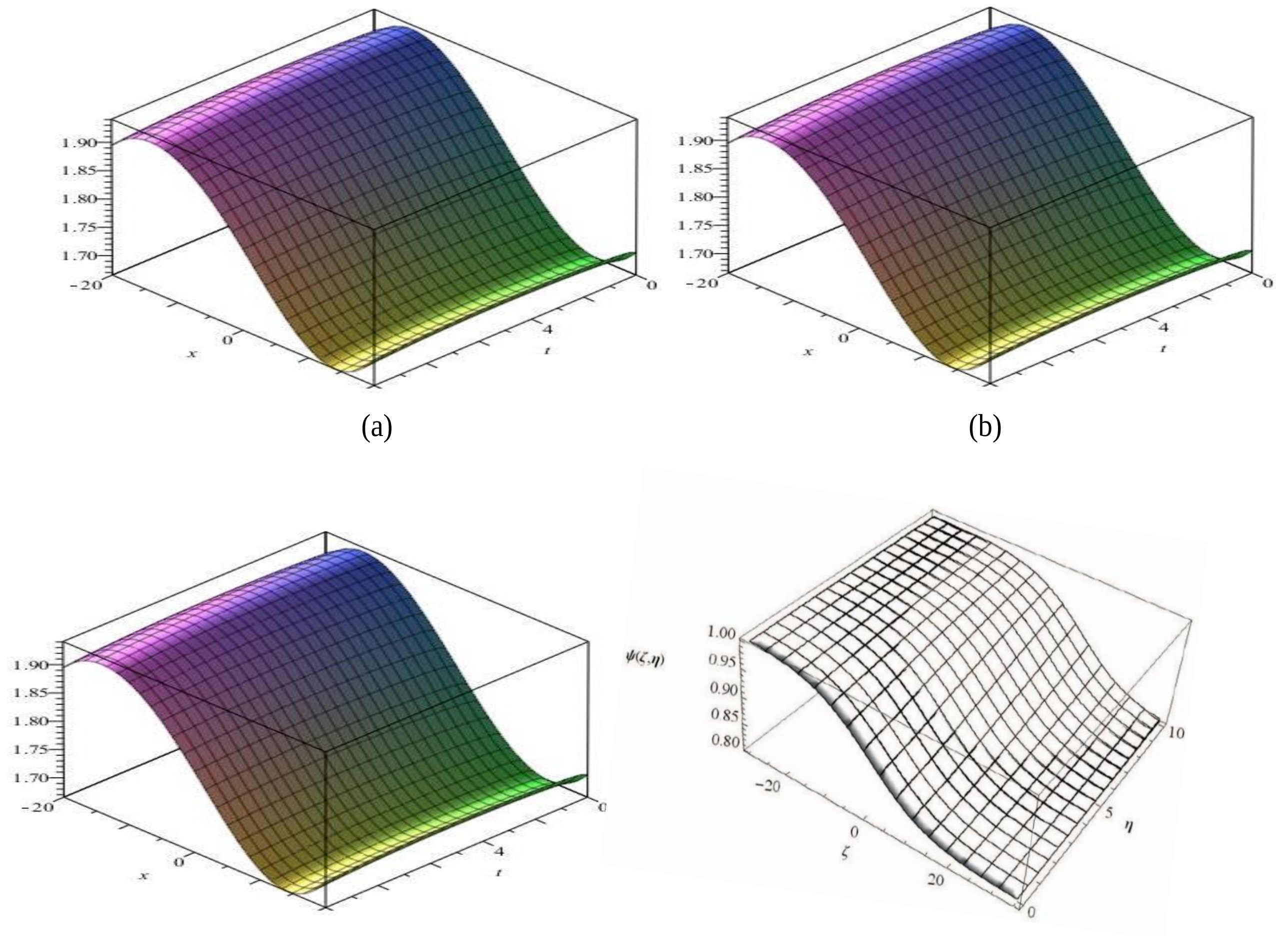
$$y_{n+1} = y_n(x,t) - \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}(D^\alpha y_n(t) - y_n'' + y_n(1-y_n)(y_n - k))\,d\tau \quad (6.6)$$

By setting $n = 0, 1, 2 \ldots$ we obtain $y_1, y_2, y_3, \cdots$ with the aid of MAPLE 18 software.
With $k = 0.8$ and $\alpha = 0.4$ we obtain the graphical results for the time fractional Nagumo equation with

$$y(x,t) = y_0(x,t) + y_1(x,t) + y_2(x,t) + \cdots \quad (6.7)$$

NUMERICAL RESULTS

Srivastava et al. (2018) while considering the use of the ADM in finding the approximate solution of the time fractional Nagumo equation whereby series of iterations were carried out to arrive at a better approximation of the exact solution, they opined that the absolute error decreases as the number of terms of the ADM solution increases. On the other hand, it was observed while employing the MVIM using the MNPs as modifier that only two iterations was needed to arrive at the approximate solution of the time fractional Nagumo equation for $\alpha = 0.4$, $\alpha = 0.6$, $\alpha = 0.8$, making it a better iterative method for the equation when compared (Figure 1).



(a) (b)

(c) (d)

Figure 1. (a) $\alpha = 0.4$, (b) $\alpha = 0.6$, (c) $\alpha = 0.8$, (d) Exact solution (Srivastava et al., 2018).

CONCLUSION

The VIM has been successfully modified by the MNPs to find the approximate solution of the time fractional Nagumo equation, whereby we presented a lemma to ascertain the boundedness of the equation. The VIM with the MNPs requires very little iteration to arrive at our results. Our results were presented in a graphical form and comparison made with the exact solution found in literature. Furthermore, we presented a comparison of our numerical values with those found in literature and discovered a minimal absolute error for the various values of α. When $\alpha = 0.4$, and $x = -10$, the error between the MVIM and the exact solution at $t = 0$ was non-existent as the solution converged absolutely, even at $\alpha = 0.6$ and 0.8. As the value of t increased, a slight error emerged which when approximated will produce an absolute convergence to the exact solution Table 1.

CONFLICT OF INTEREST

The authors have not declared any conflict of interest.

Table 1. Comparison of exact solution with MVIM solution at different values of α.

x	t	Exact $\alpha = 0.4$	MVIM $\alpha = 0.4$	Absolute error	MVIM $\alpha = 0.6$	Absolute error	MVIM $\alpha = 0.8$	Absolute error
	0	0.9007071	0.9007071	0.0000e+00	0.9007071	0.0000e+00	0.9007071	0.0000e+00
	1	0.9014141	0.9139051	1.2491e-02	0.9107579	9.3438e-03	0.9107642	9.3482-03
	2	0.9021210	0.9108235	9.4093e-03	0.9157782	1.3657e-02	0.9227768	1.3675e-02
-10	6	0.9028277	0.9174488	1.4621e-02	0.9230510	2.0823e-02	0.9227768	1.9949e-02
	10	0.9035341	0.9093486	5.8145e-03	0.9071223	3.5882-e03	0.8985340	5.0000e-03
	0	0.9014141	0.9014141	0.0000e+00	0.9014141	0.0000e+00	0.9014141	0.0000e+00
	1	0.9028277	0.9115290	8.7013e-03	50.9114635	8.6358e-03	0.9114679	8.6402e-03
	2	0.9042401	0.9146101	1.0370e-02	0.9164830	1.2243e-02	0.9165006	1.2261e-02
-5	6	0.9056508	0.9181528	1.2502e-02	0.9243537	1.8703e-02	0.9234794	1.7829e-02
	10	0.9070593	0.9100518	2.9952e-03	0.9028734	7.6406e-04	0.8992351	7.8242e-03
	0	0.8985859	0.8985859	0.0000e+00	0.8985859	0.0000e+00	0.8985859	0.0000e+00
	1	0.8971723	0.9087007	1.1528e-02	0.9086352	1.1463e-02	0.9086396	1.1467e-02
	2	0.8957599	0.9117818	1.6022e-02	0.9136547	1.7895e-02	0.9136723	1.7912e-02
5	6	0.8943492	0.9153244	2.0975e-02	0.9215253	2.7176e-02	0.9206510	2.6302e-02
	10	0.8929407	0.9072253	1.4283e-02	0.9049949	1.2054e-02	0.8964066	3.4059e-03

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