ON THE EXISTENCE OF CANONICAL FORM IN ALL SQUARE- INTEGRABLE MARTINGALE WITH RESPECT TO $\{\mathcal{F}_W(H), H \in \mathcal{M}\}$ IN THE WIENER FUNCTIONAL SPACE UNDER VERY GENERAL CONDITION ON $\mathscr{M}$

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ABSTRACT

The Wiener space is the collection of all continuous function on a given domain, taking values in a metric space, and the Wiener functional space has a canonical form of any square- integrable functional in terms of the integrals. This paper attempted to shown the existence of a canonical representation of all square-integrable Martingale with respect to $\{\mathcal{F}_W(H), H \in \mathcal{M}\}$ under very general condition on $\mathscr{M}$. The major key here is to define multiple stochastic integral of the form $\int_{\mathcal{H}^m} \tau(h_1, h_2, \dots, h_m) W(dh) \dots W(dh_m)$ where τ is (in general) a random integrand β-adapted in a suitable sense. This was achieved by critically examining a formula for changing a multiple stochastic integrals onto $\mathrm{L}^2(\eta, \mathcal{F}_\mathrm{W}\, \mathcal{A})$ and adapting an iterated formula, which will be to obtain through the application of iterated integrals.

Keywords: Canonical form, Square-integrable Martingale, random integrand, iterated formula, iterated integrals.

INTRODUCTION

In Mathematics, Wiener space is the collection of all continuous function on a given domain (usually a sub interval of the real line), taking value in metric space (usually n-dimensional space in Euclidean space). Wiener space is useful in the study of stochastic process whose sample path is continuous function. It is named after the American Mathematician Norbert Wiener for his investigations on the mathematical properties of the one-dimensional Brownian motion. Considering, $P \subseteq \mathfrak{R}^n$ and a metric space (S, d), then one can define the classical Wiener space $C_0(P; S)$ as the space of all continuous function $f: P \to S$ ie for every fixed t in P

$$d\big(f(a), f(t)\big) \to 0 \; as |\, a - t| \to 0.$$

of a uniform convergence on $[0, T]$, or uniform topology. (ii) The classical Wiener space has a separability and completeness property. It is stated here that S is both a separable and complete space; separability is a consequence of the Stone-Weiestrass theorem while completeness is a consequence of
the fact that the uniform limit of a sequence of continuous functions is itself continuous. (iii) The Wiener space is also known with the property of tightness; this is evident on the application of the Arzel$\hat{a}$-Ascoli

and only if the following conditions are met: $\lim_{a\to\infty}\lim_{n\to\infty} sup\mu_n\{f\epsilon C||f(0)|\geq a\} = 0$, and $\lim_{\delta\to 0}\lim_{n\to\infty} sup\mu_n\{f\epsilon C|\omega_f(\delta)\geq\varepsilon\} = 0$ for all $\varepsilon > 0$. Revuz and Yor (1999).

There is a standard measure on C_0 which is the Wiener measure. It is also known as a Gaussian measure which is strictly a positive probability space. However, all Gaussian measures can be represented by the abstract Wiener space transformation as stated by the structure theorem for Gaussian measures. Madras and Sezer (2011).

The theory of iterated integrals was first introduced by Chen in Frederick, et al (2015) in order to construct functions on the (infinite-dimension) space of paths on a manifold, and has since become a prominent tool in various branches of algebraic geometry, topology and number theory. The idea behind an iterated integral is closely connected to the concept of single-variable calculus. Fubini's theorem helps us to determine iterated integrals without the use of limit definition, but by taking the integral one at a time. This is prominent in the application of fundamental theorem of calculus from single – variable calculus to finding the exact value of each integral, beginning with inner integral. The theorem affirms the uniqueness and consistency of results regardless the order of integration. Mathew, et al (2022). In multivariable calculus, an iterated integral is the outcome of applying integrals to a function of more than one variable by considering some of the variables as given constants. Also we discussed a multiple integral as a function of several real variables. For example,$f(p,q)$ or $f(p,q,r)$. Integrals of a function of two variables over a region in R^2 (real-number plane) are called double integrals and integrals of a function of three variables over a region in R^3 (real-number 3 dimensional spaces) are called triple integrals according to Stewart (2008). For multiple integrals of a single-variable function, thus we consider Cauchy formula for repeated integration.

Researchers have revealed that Wiener measure and Wiener measurability acted poorly when subjected to change of scale projection and translation. It was under this prediction that Cameron and Martin,(1947) demonstrated by way of proof that an analytic Feynman integral can be written by a limit of Wiener integrals for a larger class of functional on the Wiener space which actually yield a positive result indicating a good change of scale formula for Wiener integrals on the Wiener space.

Cameron and Martin (1945) examined the behavior of Wiener integrals on projections and translations and in 1947, and established the reaction of measures and measurability within the contest of change of scale on the Wiener functional space. In a similar vain, the Banach algebra β of functional on $C_o[0,T]$ was introduced by Cameron and Storvick (1988), Deereusefond et al (1998).

The study of multiple stochastic integrals related to a class of set was carried out by Hajek and Wong (1983) where special cases of multiple Wiener integral and It$\dot{o}$ integral were analyzed. Wong - Zakai extended this result in order to obtain its generalization through specialization of the class of set adequately. Hajek and Wong (1983), constructed formulas for transforming a stochastic integral onto the space of Wiener functional and also transforming multiple stochastic integrals as iterated integrals

A new result of analytic function on X was introduced by Setsuo (2001) under the work frame of analytic functions on abstract Wiener spaces. He proved that stochastic line integrals of real analytic have 1-forms along Brownian motion, revealing also that solutions to stochastic differential equations with real analytic coefficient are analytic Wiener functional. (Horfely (2005).

The general objective of this paper is to demonstrate the existence of a canonical representation of all square- integrable Martingale with respect to $\{\mathcal{F}_W(K), K \in \mathcal{H}\}$ under very general condition on $\mathscr{M}$, While the specific objective is to; relate Multiple Stochastic Integral with all square- integrable Martingale with respect to $\{\mathcal{F}_W(H), H \in \mathcal{M}\}$ under very general condition on $\mathscr{M}$; establish a relationship between multiple stochastic integrals and iterated integrals in the Wiener functional space and demonstrate that every Wiener functional has a canonical form of any square-integrable function in terms of the integrals defining a multiple stochastic integrals.

DEFINITIONS

In this section, we shall endeavor to define some concepts as relating to the research topic and also give a clearer meaning of variables and notations according to their usage in this research work.

We shall commence this section by critically looking at the meaning and characteristics of the Wiener process as;

(Mean square integrable): A random process X is mean square-integrable from a to b if

$E\left[\int_a^b X^{\,2}(t)dt\right]$ is finite. The class of all such processes will be denoted as $\mathcal{S}_2[a, b]$.

Note that if X is bounded on $[a, b]$, in the sense that $|X(t)| \leq \mathcal{M}$ with probability 1 for all $a \leq t \leq b$, then X is square-integrable from a to b.

($\mathcal{S}_2$ norm): The norm of a process $X \in \mathcal{S}_2[a, b]$ is its root-mean-square time integral:

$$||X||_{\mathcal{S}_2} \equiv \left| E\left[\int_a^b X^2(t)dt\right] \right|^{1/2}$$

(It$\hat{o}$ integral of an elementary process): if X is an elementary, progressive, non-anticipative process, square-integrable from a to b, then its It$\hat{o}$ integral from a to b is

$\int_a^b X(t)dW \equiv \sum_{i\geq 0} X(t_i)\big(W(t_{i+1})\big) - W(t_i)$ where the t_i are the increasing sequence of time starting at 0, truncated below by a and above by b.

(It$\hat{o}$ integral): let X be progressive, non-anticipative and square-integrable on $[a, b]$. then its

It$\hat{o}$ integral is $\int_a^b X(t)dW \equiv \lim_n \int_a^b X_n(t)dW$

Taking the limit in L_2, with X_n , we say that X is It$\hat{o}$-integrable on $[a, b]$.

Stochastic Process: A stochastic process $X = \{X(t), t \in T\}$ is a collection of random variables on a common probability space $(\Omega, \mathcal{A}, \mathcal{F})$. It can also be written as a function $X: Tx\, \Omega \to \boldsymbol{R}$ such that $X(t, .)$ is $\boldsymbol{\mathcal{A}}$: Ꞁ-measurable in ω ∈ Ω for each $t \in T$. Hajek and Wong (1981).

Where $\Omega = \boldsymbol{R}^T$ as the set of all functions ω: $T \to \boldsymbol{R}$ and express $X(t, \omega) = \omega(t)$, so that ω becomes the sample path, while $\boldsymbol{\mathcal{A}}$ is the δ −algebra generated by cylinder sets having the form

$B = \{\omega \in \Omega: X(t,\omega) \in k_i for\ i = 1,2,\ldots\ldots,n\}$
where $k_i \in T$ and $k_i \in \Omega$ with assigned probability as

$$P(B) = \int_{k_1 \times k_2,\ldots,\times k_n} I_B dF_{t_1,t_2,\ldots,t_n}(T_1, T_2, \ldots, T_n) \quad (1)$$

Separable processes: A stochastic process $X = (X_t, t \in T)$ defined on a complete probability space $(\Omega, \mathcal{A}, \mathcal{F})$ is a separable process if we can obtain countably dense subset $D = \{d_1, d_2, d_3, \ldots.\}$ of T generally referred to as separant set, given that for any open and closed interval where I_z and I_{z_1} are denote open and closed interval respectively, then the subset

$\bar{B} = \cup_{t \in T \cap I_z}(\omega \in \Omega: X_t(\omega) \in I_{z_1})$ of Ω is not in comparison with the

$B = \cup_{d_j \in D \cap I_z}\left\{\omega \in \Omega : X_{d_j}(\omega) \in I_{z_1}\right\}$
which is the event.

By virtue of the law of completeness, $\bar{B}$ is an event and $P(\bar{B}) = P(B)$

Abstract Wiener space: Kuo (1972) define a set in an abstract Wiener space and examine the following stochastic integral in the set define as; $Q \subset D$ be the abstract Wiener space, then

$$X(t) = x + \int_0^t U\big(r, x(r) dW(r)\big) + \int_0^t \delta\big(r, x(r)\big) dr \quad (2)$$

Where $W(t)$ is a Wiener process in D.

Piech generated a fundamental solution $\{h_t(r, dy)\}$ which is related to the process of X_t by

$$\int f_D(g) h_t(x, dg) = E_x\big[f\big(x(t)\big)\big] \quad (3)$$

for bounded $l^p - 1$ function of f.

Covariance-type operator on Wiener space: Defining a covariance-type operator on Wiener space, Setsuo (2001), uses two random variables, J and K in the Gross-Sobolev space $M^{1,2}$ of random variable having square-integrable Malliavin derivative by letting

$$\Gamma J.K := (MJ - MP^{-1}K) \quad (4)$$

where M is the Malliavin derivative and Γ is the notion of covariance and canonical metric for vector and random fields.

Abstract Wiener space ($\boldsymbol{\mu}$) : An abstract Wiener space (μ) about the first variation for the functions in $f(D^n)$ of the form

$$K(\bar{x}) = \int_k exp\{\textstyle\sum_{i=1}^n (k, x_i)^2\} d\mu(k), \mu \in \mu(k) \quad (5)$$

Progressive Process: A continuous-parameter stochastic process X adapted to a filtration $(\mathcal{M}_t)$ is progressively measurable or progressive when $X(s,w), 0 \leq s \leq t$, is always measurable with respect to $\beta_t \times \mathcal{M}_t$ where β_t is the Borel δ-field on $[0, t]$. If X has continuous sample paths, for instance, then it is progressive.

Non-anticipating filtrations processes: Let τ be a standard wiener process, $\{\mathcal{F}_t\}$, the right-continuous completion of the natural filtration of τ, and $\mathscr{H}$ any δ-field independent of $\{\mathcal{F}_t\}$. Then the non-anticipating filtrations are the ones of the form $\delta(\mathcal{F}_t \cup \mathcal{M}), 0 \leq t \leq \infty$. A stochastic process X is non-anticipating if it is adapted to some non-anticipating filtration.

METHODOLOGY

Having critically examined related work of some researchers, it is expedient that various presentations have been made in different instances in order to establish the existence of a canonical representation of all square-integrable Martingale with respect to $\{\mathcal{F}_W(K), K \in \mathcal{H}\}$ under very general condition on $\mathscr{M}$,

However in this research, we shall be present the following as the methodology of this paper.

Multiple Wiener Integrals (MWI)

We shall define the Multiple Wiener integrals (MWI's) of the following two types

$$I_r(f) = \int \dots \dots . . \int f(h_1, \dots \dots, h_r) dW_{h_1}, \dots ., dW_{h_r} = \int f(h) dW_h^r$$
$$J_r(f) = \int \dots \dots \int f(h_1, \dots . . , h_r) W_{h_1}, \dots . , W_{h_r} dh_1, \dots . , d_{h_r} = \int f(h) X_h^r dh$$

Where $r = 1,2, \dots \dots$ while dealing with integral I_r resp. F_r, we will assume that (I) resp.(F)

(I): $W_h = 0$ a.s for some $h_0 \in \boldsymbol{\mathcal{H}}$

(J) : W is mean square continuouss

F3or W a Wiener process, f is taken to be a function in $L_2(\mathcal{H}^m, dh^m)$ and $(m^!)^{-\frac{1}{2}}$ is an isomorphism on $\hat{L}_2(\mathcal{H}^r, dh^r)$ (the Hibert space of all symmetric functions in $L_2(\mathcal{H}^m, dh^r)$) into $L_2(W)$. However in accordance to the Wiener process, it is necessary and more reasonable to expect that functions $f(h_1, \dots . . , h_r)$ of the form $\emptyset_1(h_1) \dots \emptyset_r(h_r), \emptyset \in \mathcal{F}(F_h)$ are admissible integrands and their integral is $I_r(f)$ is the iterated integral. $|(\emptyset_1) \dots . |(\emptyset_r)$ when $\emptyset_1, \dots . , \emptyset_r$ are orthogonal. This imply that $\mathcal{F}(\divideontimes^r \mathcal{H})$ is the proper class of integrands for the MWI's I_r: and similarly $\pi_2(\divideontimes^r \mathcal{H})$ is the proper Horfelt (2005) (see also Kloeden and Platen,(1991)) class of integrands for the MWI F_r.

The Existence of canonical form in all Square- integrable Martingale in Wiener Functional Space

In this section, we shall be discussing the existence of a canonical representation of all square- integrable Martingale with respect to $\{\mathcal{F}_W(H), H \in \mathcal{M}\}$ under very general condition on $\mathscr{M}$, The major key here is to define multiple stochastic integral of the form

$$\int_{\mathcal{H}^{\mathrm{m}}} \tau(\mathrm{h}_1, \mathrm{h}_2, \dots . , \mathrm{h}_{\mathrm{m}}) \mathrm{W}(\mathrm{dh}) \dots \mathrm{W}(\mathrm{dh}_{\mathrm{m}})$$

where τ is (in general) a random integrand β-adapted in a suitable sense. Hence by a careful cross examination of the multiple stochastic integral, we will establish the fact that the Wiener functional and its canonical form can be a means of representing the multiple stochastic integral.

Relating Multiple Stochastic Integral with all square- integrable Martingale with respect to $\{\mathcal{F}_W(H), H \in \mathcal{M}\}$ under very general condition on $\mathscr{M}$,

This is the main focus of this paper, which we intend to achieve this by using a collection of subsets of a fixed rectangle in $\Re^n$ as thus;

Let us define a collection of subset of a fixed rectangle $\mathcal{H}$ in $\Re^n$ to be $\mathscr{M}$. Given set $A_{1,}A_2, \dots \dots, A_n \in \Re(\mathcal{H})$ let define their support relative to $\mathscr{M}$ as the set of $\mathscr{M}$ $F_{A_1}, F_{A_{2,}}, \dots, F_{A_n} = \cap \{D: D \in \mathcal{M}\}$ and $D \cap A_1 \neq 0$ for $[\leq i \leq r)$ with the convention that if no such set D then the support is taken to be all of $\boldsymbol{\mathcal{H}}$. The intersection of all the set in $\mathscr{M}$ is the support of the empty collection of sets $(i.e\, m = 0)$ and its denoted by F. Let also assumed that the support of any collection of set $A_{1,}A_2, \dots \dots, A_m$ is contained in $\mathscr{M}$. This assumption can only be true just by only enlarging a given collection of set $\mathscr{M}$. If $h_1, h_2, \dots \dots, h_m$ are points in $\boldsymbol{\mathcal{H}}$, then their support will be written as $F_{h_1}, F_{h_2}, \dots \dots, F_{h_m}$. It therefore means that $h_1, h_2, \dots, h_m$ are $\mathscr{M}$-independent if no point is contained Jeong-Gyoo(2021) in the support of the remaining ones. For $\mathscr{M}= \{all\ closed\ sets\ in\ \mathcal{H}\}, F_{h_1}, F_{h_2}, \dots \dots, F_{h_m}$. is just $\{h_1, \dots \dots h_m\}$ so that $\mathscr{M}$- independent means distinct. Also for $\mathscr{M}= \{all\ convex\ sets\ in\ \mathcal{H}\}$, the support of n point is their convex hull and the points are $\mathscr{M}$-independent if and only if they are extreme points of their convex hull. When $\mathscr{H} \subset \Re^n_+$ and $\mathscr{M}= \{Y_h: h \in \mathcal{H}\}, Y_h$ is the closed rectangle bounded by the origin and h. Then $(F_{h_1}, F_{h_2}, \dots \dots, F_{h_m})$. is the smallest set in $\mathscr{M}$ which contains $(h_1, h_2, \dots, h_m)$ For further illustration, when $\boldsymbol{\mathcal{H}} \subset \Re^n_+$, and $\mathscr{M}$ is obtained by $\{X_h : h \in \mathcal{H}\}$ where $X_h = \{F \in \mathcal{H}: j_i \leq h_1 for\ some\ i\}$.

Then for $h_1, h_2, \dots, h_m \in \boldsymbol{\mathcal{H}}, \boldsymbol{F_{h_{1,}h_{2,\dots\dots}h_m} = \cup\, v_{hi}}$. Moreover $\mathscr{M}= \left\{\bigcup_{i=1}^n Y_{h_i}: m < +\infty\ and h_1, h_2, \dots, h_n \in \mathcal{H}\right\}$.

In this example, n points are unordered if and only if they are pair wise unordered.

Suppose $\widehat{\mathcal{H}}^m$ is the subset of $\mathscr{M}$- independent points in $\mathcal{H}^m$ then for a given collection $\mathscr{M}, \widehat{\mathcal{H}}^m$ would be meaningless for sufficiently large m. For instance if $\mathscr{M}= \{Y_h\}$ is the collection of rectangles bounded by the origin and $h \in \boldsymbol{\mathcal{H}} \subset \boldsymbol{\Re^m_+}$, then $\widehat{\mathcal{H}}^m$is empty for $m > n$. That is no more than n points can be $\mathscr{M}$- independent. For extreme cases, $\mathscr{M}= \{\mathcal{H}\}$, $\widehat{\mathcal{H}}^m$ is empty for all $m \geq 1$

Let define ε-support relative to $\mathcal{M}$ of $A_{1,}A_2, \dots \dots, A_n \in \Re^n(\mathcal{H})$ by

$F_{A_1,A_2,\dots\dots,A_n} = F_{D(\varepsilon,\varepsilon A_1),D(\varepsilon,A_2),\dots\dots,D(\varepsilon,A_n)}$ if given a subset A of $\boldsymbol{\mathcal{H}}$ defining $D(\varepsilon, A)$ as the set of points in $\boldsymbol{\mathcal{H}}$ of Euclidean distance at most ε from A for $\varepsilon > 0$ and let $F^{(.)}_{A_1,A_2\dots\dots,A_n}$ denote the union over all$\varepsilon > 0$ of ε-support of $A_1, A_2, \dots, A_n$

The ε- support of $A_1, A_2, \dots, A_n$ increases to $F^{(.)}_{A_1,A_2\dots\dots,A_n}$ as ε decreases to zero and $F^{(.)}_{A_1,A_2\dots\dots,A_n}$ is contained in the support of $A_1, A_2, \dots, A_n$ according to Jeong-Gyoo (2021). Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a fixed probability space and let $\{\mathcal{F}(A): A \in \Re(\mathcal{H})\}$ be a family of sub-δ-algebra of $\boldsymbol{\mathcal{F}}$ which is increasing in the sense that $A \subset D$ implies that $\boldsymbol{\mathcal{F}}(A) \subset \emptyset(D)$ and let $(W(A): A \in \Re(\mathcal{H}))$ be a Wiener process such that $\mathcal{F}_W(A) \subset \mathcal{F}(A)$ and $\mathcal{F}_w(A)$ is independent of $\mathcal{F}(A)$ for all A in $\Re(\mathcal{H})$ then these conditions are true.

(i) For every collection of rectangles $A_1, A_2, \dots, A_n$ such that$\prod_{i=1}^n E_i \subset \widehat{\mathcal{H}}^m\, \mu\left(A_i \cap F_{A_1,A_2,\dots\dots,A_n}\right) = 0;$

.(ii) For each $m \geq 1$, the mapping $h = (h_1, h_2, \ldots\ldots, h_m) \ldots\ldots F$ is a continuous map from $\mathcal{H}^m$ to the collection of compact sets under the Hausdorff metric: $u(A : D) = \left(\max_{x=A} \min_{y\in D} |x-y| + \max_{x\in D} \min_{y\in A} |x-y|\right)$
(iii) For every collection of rectangles $A_1, A_2, \ldots\ldots, A_n$ in , $\forall \epsilon > 0$.
$\mathcal{F}\left(F_{A_1,A_2,\ldots\ldots,A_N}\right) = f\left(F_{A_1,A_2,\ldots\ldots,A_n}\right)$ since $\mathcal{F}_w(A) \subset \mathcal{F}(\mathrm{A})$ for all H in $\Re(\mathcal{H})$, condition (iii) implies the following which we shall refer to as condition
(iv) For every collection of rectangles $A_1, A_2, \ldots\ldots, A_n$ in $\boldsymbol{\mathcal{H}}$, then $\left(F_{A_1,A_2,\ldots\ldots,A_n} - F^{(.)}_{A_1,A_2,\ldots\ldots,A_n}\right) = 0$.

If $\mathcal{F}_w(A) = \mathcal{F}(A)$ for all A, then condition (iii) and (iv) are equivalent , condition (ii) and (iii) is a continuity condition. For a $\mathscr{M}$ satisfying condition (i) to (iii), we shall now define multiple stochastic integral of order m as;

$$\emptyset = W^n = \int_{\mathcal{H}^m} \emptyset, W(dh_n)$$

For integrands $\emptyset(\omega, h), (\omega, h) \in \Omega X \mathcal{H}^m$ satisfying

(a1)$\emptyset$ is $\boldsymbol{\mathcal{F}}$ Xτ^m measurable.

(a2) for each $h \in \widehat{\mathcal{H}}^m \emptyset$, is $\mathcal{F}\left(J_g\right)$ - measurable

(a3)$\int_{\mathcal{H}^m} E\emptyset_h^2 \, dh < \infty$

The space function satisfying (a1-a3) is denoted by $L^2_\tau\left(\Omega \ X \ \widehat{\mathcal{H}}^m\right)$ for $\emptyset$ and θ in $L^2_\tau\left(\Omega \ X \ \widehat{\mathcal{H}}^m\right)$

Define $< \emptyset, \theta >= E^{\int_{\mathcal{H}^m} \emptyset_h \theta_h dh}$ and let ϕ denote the summarization of $\emptyset$, i.e
$\phi, = \frac{1}{m!} \sum_\alpha \emptyset_{m(h)!}, x(h) =$ permutation. Call $\emptyset$ atomic if $\emptyset(\omega, h) = \propto (\omega) I_A(h)$ where I_A is the

indicator function of a product of rectangles $A = \prod_{i=1}^n A_i$ such that$H \subset \widehat{\mathcal{H}}^m$.

To move further, we need to consider some propositions on Multiple Stochastic integrals that would be very useful in achieving our aim here.

(Completeness of multiple stochastic integrals): Let $\mathscr{M}$ be a collection of sets such that $\mathscr{M}$ and $\{\mathcal{F}_W(H)\}$satisfy condition (i)-(iii). Then every square – integrable $\mathcal{F}_W(\mathcal{H})$-measurable random variable Z has a representation of the form

$$Z = E[Z \ I\mathcal{F}(F)] + \sum_{n=1}^\infty Z_n o \ W^n \tag{10}$$

where $Z_n o W^n$ are stochastic integrals defined relatively to $\mathscr{M}$ and $F = \cap \{D : D \in \mathcal{M}\}$.

Proof :

The proposition is very familiar Ojo-orobosa (2018) Setsuo (2001) and Ustune (2018) in case $\mathscr{M}$ of all closed subsets of $\mathcal{H}$, so that the integrals are multiple Wiener integrals. Since by iterated integration formula any multiple Wiener integral can be represented as a sum of multiple stochastic integrals relative to the smaller class of set $\mathscr{M}$, proposition 4.1 is true in general. Ojo-orobosa (2018.

Proposition

For f in $L^2(\mathcal{H})$, define

$$\hat{f}^m(h_1, h_2, \dots . h_m) = \prod_{i=1}^{n} f(h_i) \quad (11)$$

And $W_n(f, K) = \left(\hat{f}^n o\, W^n\right)_K$ (12)

If $\mathscr{M}$ and $\{\mathcal{F}_W(K)\}$ satisfy condition (i) –(iii) , then for $K \in \mathcal{M}$,

$$W_n(f, H) = W_n(f, F \cap H) + \sum_{m=1}^{n} \binom{n}{m} \left[\hat{f}m(.)W_{n-m}(f, I)o\, W^m\right]_H \quad (13)$$

Proof :

Observed that $\hat{f}^m$ is symmetric and $\hat{f}^n(h_1, h_2, \dots h_n) = \hat{f}^m(h_1, h_2, \dots h_m)\hat{f}^{n-m}(h_{m+1}, \dots ., h_n)$
Thus, equation (4.11) for $K = \mathcal{H}$ is obtained by applying the iterated integration formula to express the multiple Wiener integral $W_n(f, \mathcal{H})$ in terms of Stochastic integrals relative to $\mathscr{M}$. Then (13) is true in general since each side is a martingale relative to $\{\mathcal{F}_W(K): K \in \mathcal{M}\}$. Ivan et al (2014)

Proposition

Let $\mathscr{M}$ and $\{\mathcal{F}(E): EE\mathbb{R}(\mathcal{H})\}$ satisfy the condition in section (4.2) , then either $f \in L^2(\mathcal{H})$ or f is a bounded function in $L^2_\propto(\eta \times \mathcal{H})$.

Define $L(f, E) = \exp\left((f\text{o}W)_E - \frac{1}{2}(f^2\text{o}\mathcal{M}_E)\right)$ (14)

where $(f^2\text{o}\tau)_E$ is the lebesgue integral of f^2 over E then for $E \in \mathscr{M}$,

$L(f, E) = L(f, F \cap E) + \sum_{n=1}^{\infty} \frac{1}{n!}[f^\infty m(.)L(f, F)\text{o}W^\mu]_E$ (15)

Proof

Suppose that $f \in L^2(\mathcal{H})$ for MWI'S ($\mathscr{M}$= all closed set) equation (14) reduces to

$$L(f, E) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} W_n(f, E) \quad (16)$$

Which is also well known for the case of general $\mathscr{M}$. Mathew et al (2022), Revuz and Yor (1999).
Using (14) in (16) we have,

$$L(f, E) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!}\left(W_n(f, F \cap E) + \sum_{m=1}^{n} \binom{n}{m} \left[\hat{f}^{\,m} W_{n-m}(f, F)\text{o}W^m\right]_E\right)$$

$$= L(f, F \cap E) + \sum_{m=1}^{\infty} \frac{1}{m!}\left[\hat{f}^m \sum_{j=0}^{\infty} \frac{1}{j!} W_j(f, F)\text{o}W^m\right]_E$$

$$= L(f, F \cap E) + \sum_{m=1}^{\infty} \frac{1}{m!}\left[\hat{f}^m L(f, F)\text{o}W^m\right]_E \quad (17)$$

Which establishes (4.13) for f in $L^2(\mathcal{H})$ Mathew Mathew et al (2022), Revuz and Yor (1999).

DISCUSSION

We examined the Multiple Wiener integral (MWI) by recalling a multiplication formula of the multiple Wiener integral which is embedded in the following lemma 3.1 which we proved by first proving the Leibniz formula whose proof follows from its finite dimensional version.

Also we define a collection of subset of a fixed rectangle $\mathcal{H}$ in $\mathfrak{R}^n$ to be $\mathscr{M}$. Given set $A_1, A_2, \ldots\ldots, A_n \in \mathfrak{R}(\mathcal{H})$ and their support relative to $\mathscr{M}$ as the set of $\mathscr{M}$

$F_{A_1}, F_{A_2}, \ldots, F_{A_n} = \cap \{D : D \in \mathcal{M}\}$ and $D \cap A_1 \neq 0$ for $[\leq i \leq r)$ with the convention that if no such set D then the support is taken to be all of $\boldsymbol{\mathcal{H}}$. The intersection of all the set in $\mathscr{M}$ is the support of the empty collection of sets $(i.e\ m = 0)$ and its denoted by F.

We also discussed we considered some prepositions and theorems such as; The isometry in nature, of multiple stochastic integral, this property can be interpreted as; Suppose for each $n \geq 1$, and $h \in \mathcal{H}^m$ then $\{\phi_{n,m}(h) : m \geq 1\}$ is said to be a complete orthogonal basis for the space of square integrable $\boldsymbol{\mathcal{F}}(F_h)$-measurable random variable, let assume that $\phi_{n,m}(h)$ is a symmetric function in h, then the multiple stochastic integral isometry property is the set of ‘’incremental’’ random variables. $[\phi_{n,m}(h) W(dh_1) W(dh_2) \ldots\ldots W(dh_m); n \geq 0, m \geq 1, h \in \mathcal{H}^m]$

This actually helped us to achieve the set goal of this paper.

Having critically examined the canonical form of all square-integrable Martingale with respect to $\{\mathcal{F}_W(K), K \in \mathcal{H}\}$ under very general condition on $\mathscr{M}$, the study identified that;

(i)every Wiener functional has a canonical form of any square-integrable function in terms of the integrals defining a multiple stochastic integrals,

(ii)demonstrate a relationship between multiple stochastic integrals and iterated integrals in the Wiener functional space and

(iii)there is a canonical representation of all square- integrable Martingale with respect to $\{\mathcal{F}_W(K), K \in \mathcal{H}\}$ under very general condition on $\mathscr{M}$

CONCLUSION

Maintaining that W is a Wiener process, we explored the relationship between the MWI's and the stochastic integrals by establishing that each MWI can be transformed and written as an iterated integral.

That is for $f_n \in L_2(\hat{\otimes}^m \mathcal{H})$

$$\int f_m(h) dW_h^m{}_{h_m} = \int_{\mathcal{H}} \left(\int_{\mathcal{H}} \left(\ldots \left(\int_{\mathcal{H}} f_m(h_1, \ldots\ldots, h_m) dW_{h_1} \right) \ldots\ldots dW_{h_{m-1}} \right) dW_{h_m} \right)$$

given in (6) where

$\mathscr{M} = \{all\ closed\ rectangles\ in\ \mathfrak{R}_+^n\ with\ the\ origin\ at\ one\ c$ Ojo-orobosa, (2018)

the iterated integral remains as defined.

Under general condition on $\mathscr{M}$, there is a canonical form of all square- integrable Martingale with respect to $\{\mathcal{F}_W(H), H \in \mathcal{M}\}$, and there after the representation for square-integrable Wiener functional which reduces to multiple Wiener integrals form

The relationship between multiple stochastic integrals and square integrable martingale was established in this paper by considering two major operations which includes the rule for which the conditional expectation of a multiple stochastic integral is obtained if given $\mathcal{F}_W$ and secondly the operation involving the application of an iterated integral method for expressing multiple stochastic integral defined relatively to $\mathscr{M}$ in terms of stochastic integrals in relation to another class of sets $\mathscr{M}$. These eventually provide the bases for relating Multiple Stochastic Integral with all square-integrable Martingale with respect to $\{\mathcal{F}_W(H), H \in \mathcal{M}\}$ under very general condition on $\mathscr{M}$,

These operations are very relevant more importantly the iterated integral method as far as this paper is concerned and also very prominent in stochastic calculus in general.

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$\mathscr{M}=$
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