SOLUTION TO FRACTIONAL ORDER ADVECTION-DISPERSION PROBLEMS USING GALERKIN METHOD WITH MAMADU-NJOSEH POLYNOMIALS

Odjeva, Joseph O.[1] and Njoseh, Ignatius N.[2]

[1,2]Department of Mathematics, Delta State University, Abraka, Nigeria.
[2]Corresponding Author's email: njoseh@delsu.edu.ng

Abstract

The advection-dispersion equation is a partial differential equation describing a probability function for the location of particles in continuum. Finding the analytic solution to this equation is very difficult and cumbersome. Thus, in this research, we have considered the numerical approximation of fractional time and space advection-dispersion equation. Specifically, Galerkin Method was adopted as the numerical method with Mamadu-Njoseh polynomials as basis functions to obtain the approximate solution of the fractional adventure-dispersion equation. The study established that the Galerkin method effectively solves fractional order advection-dispersion equation with time and space derivatives and that the method converges rapidly with an increase in the value of the fractional order α, for $t = 0.1$. The numerical results obtained show that the method converges rapidly to the exact solution.

Keywords: Fractional derivative, Galerkin Method, Mamadu-Njoseh Polynomials (MNPs), Algebraic Equations.

Introduction

Liu *et al* (2003) found that the advection-dispersion equation (ADE) is a common equation used to describe solute transport in aquifers. This equation is a deterministic homogeneous equation that describes a probability function for particle location in a continuum. Fractional calculus has gained interest due to its ability to accurately depict natural physical phenomena and dynamic system processes. FDEs are useful in modeling various systems in physics, chemistry, and engineering, such as viscoelastic systems, dielectric polarization, chaotic behavior, control theory, and electrolyte-electrolyte polarization

(Lazopoulos, 2006 and Chen *et al.,* 2015). The study of FDEs has led to significant focus on the exact and numerical solution of fractional differential equations and integral equations. Numerous schemes, methods, and treatments have been proposed to obtain the numerical solution of FODEs in recent literature.

Meerschaert and Tadjeran (2004) devised finite difference approximations to solve one-dimensional fractional advection–dispersion equations with Dirichlet boundary conditions in the context of fractional order advection dispersion equations. An algorithm based on the theorem was proposed by Jiang and Lin (2010) to solve fractional advection–dispersion equations. In order to solve fractional advection–dispersion equations with fractional derivative boundary conditions, Liu and Hou (2017) devised an implicit finite difference approach. Additionally, the authors provided evidence of the methodology's first-order convergence, solvability, unconditional stability, and consistency. In anomalous diffusion, Zhang *et al*; (2017) found nontrivial solutions to a fractional advection–dispersion equation. A finite difference approach to solving the Riesz fractional advection–dispersion equations was presented by Zhang (2018). An asymmetric discretization method and a modification of the shifted Grunwald approximation to the fractional context were used to arrive at the solution. A finite element method was proposed by Roop (2006) to find numerical solutions of the two-dimensional fractional advection-dispersion equation. Shen *et al*; (2014) used the Riesz operator in conjunction with the fractional finite difference approximation schema with a weighting factor for the fractional advection-dispersion equation. In order to solve the two-sided Fractional Advection-Dispersion Equation, Bhrawy *et al*; (2015) presented a method based on the operational matrices. Using the Caputo Fractional Reduced Differential Transform Method (CFRDTM), Fadugba *et al*; (2021) carried out fractional numerical research on the

Advection-Dispersion Equation (ADE) with Fractional Order (FO). The Caputo Fractional Derivative (CFD) and the well-known Transform Method (RDTM) are combined to create CFRDTM. Using CFRDTM, a convergent series solution for ADE with FO is obtained.

In this research, our concern is to seek the numerical solution of the fraction order advection-dispersion equation with time- and space-fractional derivatives of the form:

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = -\upsilon \frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}} + k \frac{\partial^{2\beta} u(x,t)}{\partial x^{2\beta}} + f(x,t), t > 0, \quad x > 0, \quad 0 < \alpha \leq 1, 0 < \beta \leq 1 \tag{1}$$

with initial conditions as

$$u(0,t) = f_1(t), u_x(0,t) = f_2(t), \tag{2}$$

and boundary conditions

$$u(x,0) = g(x), \tag{3}$$

where u is solute concentration, the positive constants υ, k represent the average fluid velocity and the dispersion coefficient, x is the spatial domain, t is time, and α and β are parameters describing the order of the time- and space-fractional derivatives, respectively and $f(x,t)$ is a source/sink term. The fractional derivatives are considered in the Caputo sense. Variables in the overall reaction expression specify which fractional derivatives to change in order to produce different reactions. The fractional equation becomes the traditional advection-dispersion equation (ADE) when $\alpha = \beta = 1$. The macroscopic transport coefficients, υ and κ, determine the mean and variances of Gaussian density solutions, which will be the main solutions of the ADE over time. Using Mamadu-Njoseh polynomials as the basis, the Galerkin method will be used to derive the numerical solution of FOADEs.

Caputo's differential operator

Definition 1. Suppose that $\alpha > 0, x > 0, \alpha, x \in R$. the fractional operator

$$D_*^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(\mathrm{n}-\alpha)} \int_0^x \frac{f^{(n)}(s)\, ds}{(x-s)^{\alpha+1-n}}, & n-1 < \alpha < n \,\epsilon\, N, \\ \frac{d^n}{dx^n} f(x), & \alpha = n \in N, \end{cases} \tag{4}$$

is called the Caputo fractional derivative or Caputo fractional differential operator of order α

Definition 2: The Riemann-Liouville Fractional Derivative Operator (RLFDO) of $f\ (t)$ is given by

$${}_0D_t^\alpha f(t) = \frac{1}{\Gamma(\mathrm{n}-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau, \tau > 0, n-1 < \alpha < n, n \in N \tag{5}$$

Definition 3: The Caputo Fractional Derivative (CFD) of $f(t) \in C_{-1}^n, n \in N$ is given by

$${}_0^cD_t^\alpha f(t) = \frac{1}{\Gamma(\mathrm{n}-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \ \text{ for } \alpha \in (n-1, n], t > 0 \tag{6}$$

Definition 4: The Caputo Time-Fractional Derivative Operator (CTFDO) of order $\alpha > 0$ is as follows

$${}_0^cD_t^\alpha u = \begin{cases} \frac{1}{\Gamma(\mathrm{n}-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} u^{(n)}(x,\tau) d\tau & \alpha \in (0,1], \\ \frac{\partial^n \mathrm{u(x,t)}}{\partial t^n}, & \alpha = n \end{cases} \tag{7}$$

where n is the smallest integer that exceeds α, $u = u(x,t)$ and $u^{(n)}(x,\tau) = \frac{\partial^n \mathrm{u(x,t)}}{\partial t^n}$

In the Caputo fractional differential equation, initial conditions have clear physical interpretation which is the main advantage of CFDO over RLFO.

Definition 5: Let the function $h(x)$ be differentiable and let α be the order of the derivative. Then the caputo operator of the fractional derivative can be defined as follows (Sweillam and Khader, 2010):

$$D^\alpha h(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x h^{(m)}\,(t)(x-t)^{m-(\alpha+1)} dt, \ \ \alpha > 0, x > 0, \tag{8}$$

where $m-1 < \alpha \le m,\ m\ \in \mathbb{N}$.

The Caputo operator is linear. Moreover, using Definition 4 to obtain the fractional derivative of the constant function K and polynomials x^m we have

$$D^\alpha K = 0, K \text{ is a constant.} \tag{9}$$

$$D^\beta x^m = \begin{cases} 0, & m \in \{0,1,2,\dots,\lceil\alpha\rceil - 1\}, \\ \frac{\Gamma(\mathrm{m}+1)}{\Gamma(\mathrm{m}+1-\alpha)} x^{\mathrm{m}-\alpha}, & \mathrm{m} \in \mathbb{N} \wedge \mathrm{m} \ge \lceil\alpha\rceil \text{ or } \mathrm{m} \notin \mathbb{N} \wedge \mathrm{m} > \lceil\alpha\rceil - 1 \end{cases} \tag{10}$$

where the ceiling function of τ is $\lceil\tau\rceil$

The Orthogonal Polynomials

The orthogonal polynomials are class of polynomial $p_n(x)$ define over a range $[a,b]$ that obeys the orthogonality relation (Mamadu and Njoseh, 2016)

Let $\int_a^b w(x)\varphi_i(x)\varphi_j(x)dx = h_i\delta_{ij}$ (11)

where δ_{ij} is the Kronecker delta denoted by $\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$

and the weight function $w(x)$ is continuous and positive on $[a, b]$ such that the moments

$$u = \int_a^b w(x)x^i dx, \quad i = 0,1,2,3,\dots \tag{12}$$

exist.

Thus the integral,

$$\langle\varphi_i,\varphi_j\rangle = \int_a^b \mathrm{w}(x)\varphi_i(x)\varphi_j(x)dx \tag{13}$$

is called the inner product of the polynomials φi and φj, with the orthogonality property,

$$\langle\varphi_i,\varphi_j\rangle = \int_a^b \mathrm{w}(x)\varphi_i(x)\varphi_j(x)dx = 0, \ \ \mathrm{i} \neq \mathrm{j}, \mathrm{x} \in [-1,1] \tag{14}$$

$If\ \delta_{ij} = 1$, then the polynomials are not only orthogonal but orthonormal. Hence, we adopt the weight function $w(x) = x^2 + 1$ in the interval $[a,b] \equiv [-1,1]$

The construction of φ_i, $i = 1,2,3,\dots$ of the approximant:

$$\tilde{y}(x) = \sum_i^a a_i\varphi_i(x) \cong y(x) \tag{15}$$

then follows.

Mamadu-Njoseh Polynomials (Njoseh and Mamadu, 2016a; 2016b; 2017a and 2017b)

These are orthogonal polynomials constructed with respect to the weight function, $w(x) = x^2 + 1, x \in [-1,1]$ using the three properties:

i. $\varphi_n(x) = \sum_{i=0}^n C_i^{(n)} x^i$,

ii. $< \varphi_m(x), \varphi_n(x) >= 0, m \neq n$,

iii. $\varphi_n(x) = 1$.

where $\varphi_i, i = 0,1,2,3,\dots$

Mamadu-Njoseh Polynomials: Definition and Properties in Fractional Sense

The Mamadu-Njoseh polynomials $\{\varphi_n^{(\alpha)}(x)\}_{n=0}^{\infty}, \alpha \geq 0$, in the fractional sense, can be constructed using the following properties (Njoseh and Mamadu, 2016a; Mamadu and Njoseh, 2016; Ogeh and Njoseh, 2019)

$$\varphi_n^{(\alpha)}(x) = \sum_{r=0}^{n} C_r^{(n)} \frac{x^r}{\Gamma(\mathrm{r}\alpha+1)}, \qquad (16)$$

$$\langle \varphi_m^{(\alpha)}(x), \varphi_n^{(\alpha)}(x) \rangle = 0, m \neq n \qquad (17)$$

$$\varphi_n^{(\alpha)}(x) = 1 + \alpha - x \qquad (18)$$

These polynomials are orthogonal concerning the weight function $w(x) = \frac{x^r(1+x^2)}{\Gamma(\mathrm{r}\alpha+1)}$.

The orthogonality relation is given as

$$\frac{1}{\Gamma(\mathrm{r}\alpha+1)} \int_{-1}^{1} \frac{x^r(1+x^2)}{\Gamma(\mathrm{r}\alpha+1)} \varphi_m^{(\alpha)}(x), \varphi_n^{(\alpha)}(x) dx = \binom{r+\alpha}{r} \delta_{mn}, \qquad (19)$$

where δ_{mn} is the kronecter delta given as

$$\delta_{mn} = \begin{cases} 0 \\ 1 \end{cases}$$

The polynomials also satisfy

$$D^{\alpha} \varphi_n^{(\alpha)}(x) = (-1)^a \varphi_{n-a}^{(\alpha+a)}(x), a = 0(1)n. \qquad (20)$$

Let $u(x) \in L_w^2[-1,1]$ such that it is integrable on $[-1,1]$ with weight function $w(x)$, then it can be expressed in the series form

$$u(x) = \sum_{r=0}^{N} a_r \, \varphi_r^{(\alpha)}(x), \qquad (21)$$

where

$$\frac{\Gamma(\mathrm{r}\alpha+1)}{\Gamma(\mathrm{r}\alpha+1+\alpha)} \int_{-1}^{1} x^r(1+x^2) \, \varphi_m^{(\alpha)}(x), \varphi_n^{(\alpha)}(x) dx, r = 0,1,2, \ldots$$

If we consider the first $(n+1)$ Mamadu-Njoseh polynomials, we can write

$$u_N(x) \cong \sum_{r=0}^{N} a_r \, \varphi_r^{(\alpha)}(x) \qquad (22)$$

The Approximate Formula of the Caputo Fractional Derivative of $\varphi_n^{(\alpha)}(x)$.

The main objective of this section is to propose relevant theorems to derive a precise approximate formula of the Caputo fractional derivative of the Mamadu-Njoseh polynomials.

Lemma 1. (Mamadu *et al*; 2021 and Mamadu *et al*; 2022)

Let $\varphi_n^{(\alpha)}(x)$be given, then

$$D^{a}\varphi_n^{(\alpha)}(x) = 0, n = 0,1,2,\dots,([\alpha]-1), \alpha > 0. \quad (23)$$

Proof. It follows directly from implementing the properties of the Caputo fractional derivative (9) and (10)

Theorem 1. **(**Mamadu *et al*; 2021 and Mamadu *et al*; 2022)

Suppose $\alpha > 0$ and $u(x)$ being the analytic solution approximated by the Mamadu-Njoseh polynomials as (16) then, its Caputo fractional derivative has the form

$$D^{a}(u_N(x)) \cong \sum_{i=[\alpha]}^{N}\sum_{m=[\alpha]}^{i} a_m w_{i,m}^{(\alpha)} x^{m-\alpha}, \quad (24)$$

where

$$w_{i,m}^{(\alpha)} = \frac{1}{\Gamma(\mathrm{m\alpha}+1-\mathrm{\alpha})}\binom{i+\alpha}{i-m}.$$

Proof. Since the linear operation is valid for Caputo fractional differentiation, we have that

$$D^{\alpha}(u_N(x)) \cong \sum_{r=0}^{N} a_r D^{\alpha}\, \varphi_r^{(\alpha)}(x). \quad (25)$$

By lemma 1, we have that

$$D^{a}\varphi_n^{(\alpha)}(x) = 0, \qquad n = 0,1,2,\dots,([\alpha]-1), \alpha > 0.$$

Thus, for $n = [\alpha]$, ...N, by using (9) and(10)on (20), we have

$$D^{a}\varphi_n^{(\alpha)}(x) = \sum_{m=[\alpha]}^{i} \frac{(-1)}{\Gamma(\mathrm{m\alpha}+1-\mathrm{\alpha})}\binom{i+\alpha}{i-m} x^{m-\alpha} \quad (26)$$

Using (25) and (26) leads to the required results.

Numerical Solution of Fractional Order Advection-Dispersion Equation

Our concern in this work is to consider the numerical solution of the fractional order advection-dispersion equation with time- and space-fractional derivatives of the form

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = -v\frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}} + k\frac{\partial^{2\beta}(x,t)}{\partial x^{2\beta}} + f(x,t),\ \ t > 0,\ \ x > 0,\ \ \ 0 < \alpha \leq 1, 0 < \beta \leq 1 \tag{27}$$

with initial conditions as

$$\begin{gathered} u(0,t) = f_1(t) \\ u_x(0,t) = f_2(t) \\ u(x,0) = g(x) \end{gathered} \tag{28}$$

Using the proposed method, we first approximate the power series solution for (27) as,

$$y_n(x) = \sum_{i=0}^{n} a_r \varphi_r(x) \tag{29}$$

where $a_r, r = 1,2, \ldots, n$, are constants to be determined, $\varphi_r(x), r = 1,2, \ldots, n$ are the Mamadu-Njoseh polynomial.

The implementation of the Galerkin method is aided by the following steps:

i. Substitute (29) into (27) to obtain

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}\left(\sum_{i=0}^{n} a_r \varphi_n(x)\right) = -v\frac{\partial^{\beta}}{\partial x^{\beta}}\left(\sum_{i=0}^{n} a_r \varphi_n(x)\right) + k\frac{\partial^{2\beta}}{\partial x^{2\beta}}\left(\sum_{i=0}^{n} a_r \varphi_n(x)\right) + f(x,t), \tag{30}$$

ii. Multiply both side of (30) by $\varphi_j(x),\ j = 0,1,2, \ldots, n$, and integrate within the interval $[a, b]$ with respect to x, that is,

$$\int_a^b \left(\frac{\partial^{\alpha}}{\partial t^{\alpha}}\left(\sum_{i=0}^{n} a_r \varphi_n(x)\right)\right)\varphi_j(x)dx$$

$$= \int_a^b \left(-v\frac{\partial^{\beta}}{\partial x^{\beta}}\left(\sum_{i=0}^{n} a_r \varphi_n(x)\right) + \frac{\partial^{2\beta}}{\partial x^{2\beta}}\left(\sum_{i=0}^{n} a_r \varphi_n(x)\right) + f(x,t)\right)\varphi_j(x)dx \tag{31}$$

iii. Write equation (31) in the matrix form

$$A\underline{X} = \underline{b}x \tag{32}$$

where

$$A = a_{ij} = \int_a^b \left(\frac{\partial^\alpha}{\partial t^\alpha}\left(\sum_{i=0}^{n} a_r \varphi_n(x)\right) + v\frac{\partial^\beta}{\partial x^\beta}\left(\sum_{i=0}^{n} a_r \varphi_n(x)\right) - \frac{\partial^{2\beta}}{\partial x^{2\beta}}\left(\sum_{i=0}^{n} a_r \varphi_n(x)\right)\right)\varphi_j(x)dx$$

$$\underline{x} = x_i = (x_1, x_2, x_3, \dots, x_n)^T$$

$$b = b_j = \int_a^b f(x,t)\,\varphi_j(x)dx,\;\; j = 0(1)n$$

Solve the above system (32) using the Gaussian elimination method to obtain value of $a_i, i = 0(1)n$.

Substitute the values of the $a_i, i = 0(1)n$, into (29) when $\alpha = 2, \beta = 1$ to obtain the approximate solution.

Numerical Examples

We present some numerical computation for the solution of fractional order advection-dispersion equation with time- and space-fractional derivatives using the Galerkin method with Mamadu-Njoseh basis functions. The resulting numerical solution obtained with the method are then compared with the exact solution (as available in literature), these are presented in tables and graphs for convergence interpretation.

The error formulation for each of these problems is defined explicitly as

$$\mathrm{u_r} = |\mathrm{u(x,t)} - \mathrm{u_r(x,t)}|,\;\; \mathrm{r} = 1,2,3, \dots, \tag{33}$$

where $\mathrm{u(x,t)}$ is the analytic or exact solution, and $\mathrm{u_r(x,t)}$ is the computed solution using the modified variational iterative scheme.

Example 1

Consider the fraction order advection-dispersion equation with time- and space-fractional derivatives of the form

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^\beta u(x,t)}{\partial x^\beta} - \frac{\partial^{2\beta} u(x,t)}{\partial x^{2\beta}} + x - 2\mathrm{t}, t > 0,\;\; x > 0\,, \tag{34}$$

$$u(0,t) = 2t^2, u_x(0,t) = x - t, u(x,0) = 0,$$

with exact solution $u(x,t) = x^2 e^{-2t}$.

Following the steps in Section 6.0, we obtain the computational results in **Table1.**

Table1: Table of Results obtained for Example1 for $t = 0.1$.

x	Exact Solution	Galerkin Method $\alpha = \beta = 0.5$	Galerkin Method $\alpha = 0.5, \beta = 0.6$	Galerkin Error $\alpha = \beta = 0.5$	Galerkin Error $\alpha = 0.5, \beta = 0.6$
0	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000
0.1	0.0081873075	0.0081866667	0.0081871064	6.40864×10^{-7}	2.0111×10^{-7}
0.2	0.0327492301	0.0327466667	0.0327491112	2.56345×10^{-6}	1.189×10^{-7}
0.3	0.073.6857678	0.0736800000	0.0736856655	5.76778×10^{-6}	1.0224×10^{-7}
0.4	0.1309969205	0.1309866667	0.1309957114	1.02538×10^{-5}	1.2091×10^{-6}
0.5	0.2046826883	0.2046666670	0.2046805662	1.60216×10^{-5}	2.1221×10^{-6}
0.6	0.2947430711	0.2947200000	0.2947411520	2.30711×10^{-5}	1.9189×10^{-6}
0.7	0.4011780690	0.4011466667	0.4011751450	3.14023×10^{-5}	2.924×10^{-6}
0.8	0.5239876820	0.5239466667	0.5239854300	4.10153×10^{-5}	2.252×10^{-6}
0.9	0.6631719100	0.6631200000	0.6631705200	5.190999×10^{-5}	1.39×10^{-6}
1.0	0.8187307531	0.8186666667	0.8187305322	6.40864×10^{-5}	2.209×10^{-7}

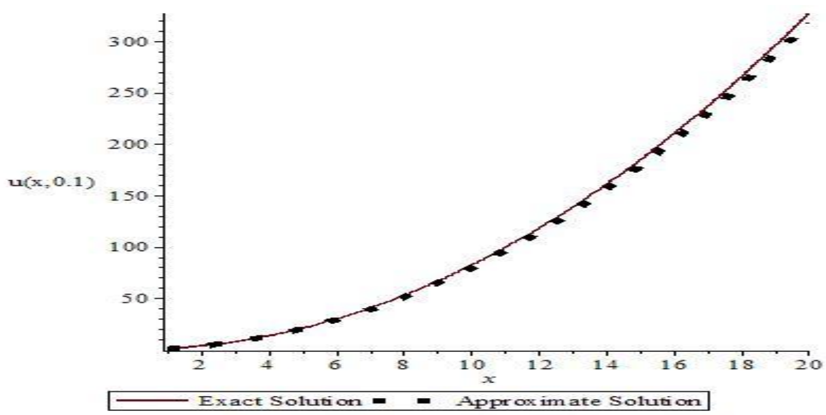

Figure 1: Comparison of Exact and Approximate Solutions for Example 1 at $\alpha = \beta = 0.5$.

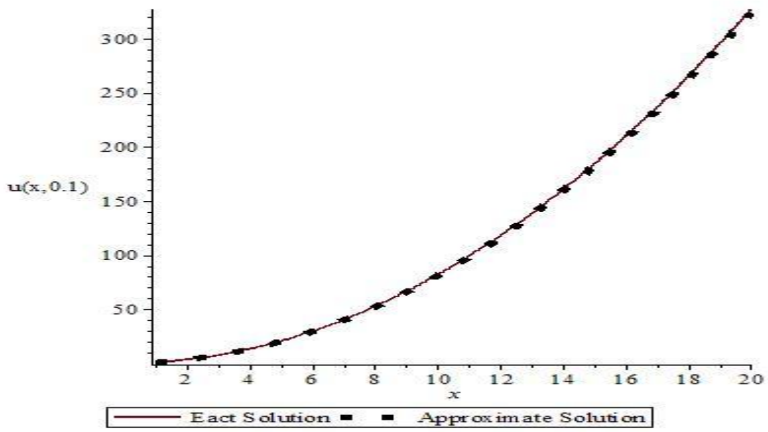

Figure 2: Comparison of Exact and Approximate Solutions for Example 1 at $\alpha = 0.5, \beta = 0.6$.

Example 2

Consider the fraction order advection-dispersion equation with time- and space-fractional derivatives of the form

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = -2.5 \frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}} - \frac{\partial^{2\beta} u(x,t)}{\partial x^{2\beta}} + xt, t > 0, \quad x > 0 \, , \tag{35}$$

$$u(0,t) = t, u_x(0,t) = 1, u(x,0) = 1.$$

The exact solution is

$$u(x,t) = \frac{x}{1-t}.$$

Following the steps in Section 6.0, we obtain the computational results in **Table 2.**

Table 2: Table of Results obtained for Example 2 for $t = 0.1$.

x	**Exact Solution**	**Galerkin Method** $\alpha = \beta = 0.5$	**Galerkin Method** $\alpha = 0.5, \beta = 0.6$	**Galerkin Error** $\alpha = \beta = 0.5$	**Galerkin Error** $\alpha = 0.5, \beta = 0.6$

0.0	0.0000000000	0.0000	0.0000000000	0.0000000000	0.0000000000
0.1	0.1111111111	0.0900	0.0960236464	0.0211111111	0.0150874647
0.2	0.2222222222	0.1800	0.1920472928	0.0422222222	0.0301749294
0.3	0.3333333333	0.2700	0.2880709391	0.0633333333	0.0452623942
0.4	0.4444444444	0.3600	0.3840945855	0.0844444444	0.0603498589
0.5	0.5555555555	0.4500	0.4801182319	0.1055555556	0.0754373237
0.6	0.6666666667	0.5400	0.5761418783	0.1266666667	0.0905247884
0.7	0.7777777778	0.6300	0.6721655247	0.1477777778	0.1056122531
0.8	0.8888888889	0.7200	0.7681891710	0.1688888889	0.1206997180
0.9	1.0000000000	0.8100	0.8642128174	0.1900000000	0.1357871826
1.0	1.1111111111	0.9000	0.9602364638	0.2111111111	0.1508746473

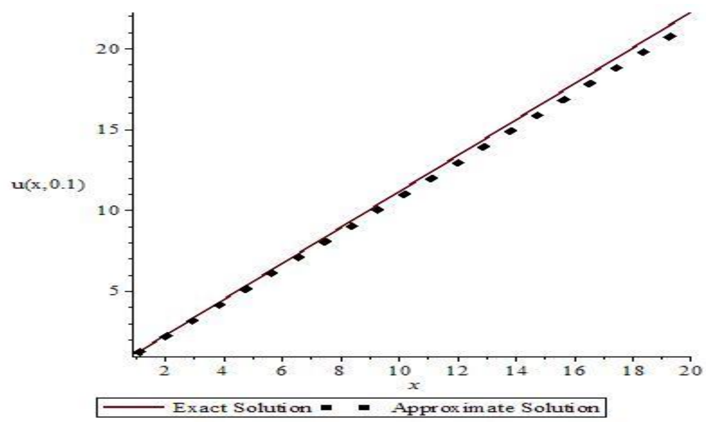

Figure 3: Comparison of Exact and Approximate Solutions for Example 2 at $\alpha = \beta = 0.5$.

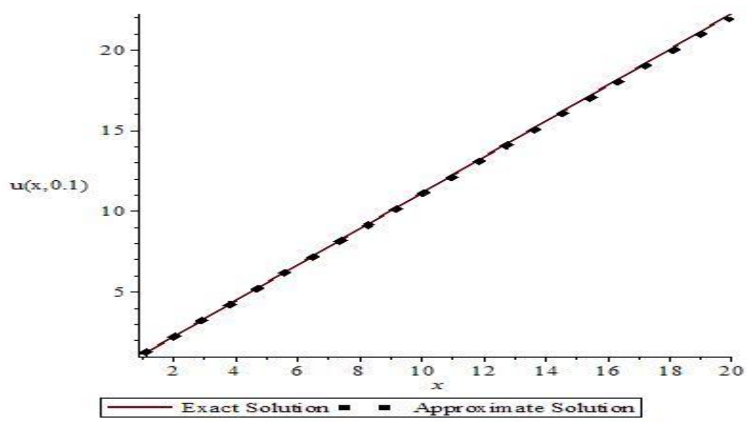

Figure 4: Comparison of Exact and Approximate Solutions for Example 2 at $\alpha = 0.5, \beta = 0.6$.

Convergence Analysis

Theorem 4.1 (Mamadu and Njoseh, 2016)

Let

$$F(u) = \frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} + v\frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}} - k\frac{\partial^{2\beta} u(x,t)}{\partial x^{2\beta}} - f(x,t), t > 0, \;\; x > 0, \quad 0 < \alpha, \beta \leq 1.$$

Then the proposed method convergences if the following conditions are satisfied:

i. $(F(u) - F(v), u - v) \geq k\|u - v\|^2, \; k > 0, u, v \in H, H\ is\ an\ Hilbert\ space.$

ii. For a $> 0, \exists\ g(\mathrm{a}) > 0$ such that $\|u\| \leq \mathrm{a}, \|v\| \leq \mathrm{a}, u, v \in H$ then

$$(F(u) - F(v), u - v) \geq g(\mathrm{a})\|u - v\|\|b\|, \;\; b \in H.$$

Proof:

For $k > 0, u, v \in H$, we have

$$(F(u) - F(v), u - v)$$

$$= \left(\left(\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} + v\frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}} - k\frac{\partial^{2\beta} u(x,t)}{\partial x^{2\beta}} - f(x,t)\right)\right.$$

$$\left. - \left(\frac{\partial^{\alpha} v(x,t)}{\partial t^{\alpha}} + v\frac{\partial^{\beta} v(x,t)}{\partial x^{\beta}} - k\frac{\partial^{2\beta} v(x,t)}{\partial x^{2\beta}} - f(x,t)\right)\right), u - v)$$

where

$$F(v) = \left(\frac{\partial^{\alpha} v(x,t)}{\partial t^{\alpha}} + v\frac{\partial^{\beta} v(x,t)}{\partial x^{\beta}} - k\frac{\partial^{2\beta} v(x,t)}{\partial x^{2\beta}} - f(x,t)\right).$$

Applying the Schwartz inequality, we get

$$\left(\left(\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} + v\frac{\partial^{\beta}u(x,t)}{\partial x^{\beta}} - k\frac{\partial^{2\beta}u(x,t)}{\partial x^{2\beta}} - f(x,t)\right)\right.$$

$$\left. - \left(\frac{\partial^{\alpha}v(x,t)}{\partial t^{\alpha}} + v\frac{\partial^{\beta}v(x,t)}{\partial x^{\beta}} - k\frac{\partial^{2\beta}v(x,t)}{\partial x^{2\beta}} - f(x,t)\right), u - v\right)$$

$$\leq k_1 \left\|\left(\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} + v\frac{\partial^{\beta}u(x,t)}{\partial x^{\beta}} - k\frac{\partial^{2\beta}u(x,t)}{\partial x^{2\beta}} - f(x,t)\right)\right.$$

$$\left. - \left(\frac{\partial^{\alpha}v(x,t)}{\partial t^{\alpha}} + v\frac{\partial^{\beta}v(x,t)}{\partial x^{\beta}} - k\frac{\partial^{2\beta}v(x,t)}{\partial x^{2\beta}} - f(x,t)\right)\right\| \|u - v\|.$$

Using the mean value theorem we obtain

$$\left(\left(\left(\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} + v\frac{\partial^{\beta}u(x,t)}{\partial x^{\beta}} - k\frac{\partial^{2\beta}u(x,t)}{\partial x^{2\beta}} - f(x,t)\right) - \left(\frac{\partial^{\alpha}v(x,t)}{\partial t^{\alpha}} + v\frac{\partial^{\beta}v(x,t)}{\partial x^{\beta}} - k\frac{\partial^{2\beta}v(x,t)}{\partial x^{2\beta}} - f(x,t)\right)\right)\right) \geq$$

$\varepsilon\|u - v\|^2$.

where $\varepsilon = k_1 \mathrm{a}^2$

Hence,

$$(F(u) - F(v), u - v) \geq k\|u - v\|^2$$

holds with $\tau = k_1 \mathrm{a}^2$.

Also, for a $> 0, \exists\, g(a) > 0$ such that $\|u\| \leq \Omega, \|v\| \leq \Omega, u, v \in H$, then

$$(F(u) - F(v), b)$$

$$= \left(\left(\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} + v\frac{\partial^{\beta}u(x,t)}{\partial x^{\beta}} - k\frac{\partial^{2\beta}u(x,t)}{\partial x^{2\beta}} - f(x,t)\right)\right.$$

$$\left. - \left(\frac{\partial^{\alpha}v(x,t)}{\partial t^{\alpha}} + v\frac{\partial^{\beta}v(x,t)}{\partial x^{\beta}} - k\frac{\partial^{2\beta}v(x,t)}{\partial x^{2\beta}} - f(x,t)\right)\right)$$

$$\leq \mathrm{k}^2\|u - v\|\|b\|\, g(a)\|u - v\|\|b\|,$$

which satisfies the second condition.

Discussion of Results

The independent variable t was fixed at $t = 0.1$ and values where obtained for the Galerkin method with Mamadu-Njoseh basis functions for $\alpha = \beta = 0.5$, and $= 0.5$, $\beta = 0.6$, in the **Tables1** and **2** respectively. The value of x was taken between 0 and 1 (with steps of 0.1). For each value of x considered, Galerkin method with $\alpha = \beta = 0.5$ gave less error when compared with same Galerkin method with $\alpha = 0.5$, $\beta = 0.6$ taking the exact solution as benchmark. In the **Table 1**, it was generally observed that Galerkin method $\alpha = \beta = 0.5$ posed an average error of $2.46732584 \times 10^{-5}$ over the interval considered, and Galerkin method with $\alpha = 0.5$, $\beta = 0.6$ posed an average error of 1.020725×10^{-6} over the same interval. Furthermore, in **Table 2,** Galerkin method with $\alpha = \beta = 0.6$generated an average error of 0.08298105603 and Galerkin method with $\alpha = 0.5$, $\beta = 0.6$ and average error of 0.1161111111 within the same interval. Thus, in both cases the Galerkin method proved to be accurate and converges significantly for the models considered.

Conclusion

It is important to note that numerical methods are used to resolve all mathematical formulations or constructions since many known analytic methods are difficult to resolve in real sense. Thus, our results have shown that the new iterative scheme Galerkin method with Mamadu-Njoseh basis functions encourages rapid convergence for the fractional order advection-dispersion equation with time- and space- derivatives. On the basis of our analysis and computation we strongly advocate that the Galerkin method with Mamadu-Njoseh basis functions as decomposer of nonlinear terms in partial differential equations, and any other mathematical equation be encouraged as a numerical method.

References

Bhrawy, A.H., Zaky, M.A., and Machado, J.A.T., (2015). Efficient Legendre spectral Tau algorithm for solving the two-sided space-time Caputo fractional advection-dispersion equation, *Journal of* Vibration *and* Control,22(8):1–16.https://doi.org/10.1177/1077546314566835.

Chen, Y., Ke, X. and Wei, Y. (2015). Numerical algorithm to solve system of nonlinear fractional differential equations based on wavelets method and the error analysis. *Applied Mathematics and Computation*, 251:475-488.

Fadugba, S.E., Edogbanya, H.O., Ogunyebi, S.N., Babalola, B.T. and Okunlola J.T. (2021). A fractional numerical study on advection-dispersion equation with fractional order. *Journal of physics conference series.* 1734 012004. DOI 10.1088/1742-6596/1734/1/012004

Jiang, W. and Lin, Y. (2010). Approximate solution of the fractional advection–dispersion equation. *Computer Physics Communications,* 181:557–561.

Lazopoulos, K.A. (2006). Nonlocal continuum mechanics and fractional calculus. *Mechanics Research Communications*, 33(6): 753-757

Liu, F., Turner, V., Anh, V. and Su, N. (2003). A two-dimensional finite volume method for transient simulation of time-, scale-and density-dependent transport in heterogeneous aquifer systems. *The Korean Journal of Computational and Applied Mathematics*, 11:215–241.

Liu, T. and Hou, M. (2017). A fast implicit finite difference method for fractional advection-dispersion equations with fractional derivative boundary conditions. *Advances in Mathematical Physics*, 1–8. https://doi.org/10.1155/2017/8716752

Mamadu, E.J. and Njoseh, I.N. (2016). Certain orthogonal polynomials in orthogonal collocation methods of solving Integro-differential equations (fides). *Transactions of the Nigeria Association of Mathematical Physics*, 2:59-64

Mamadu, E.J, Njoseh, I.N. Ojarikre, H.I. (2022). Space discretization of time-fractional telegraph equation with Mamadu-Njoseh basis functions. *Applied Mathematics*, 13(09): 760-773

Mamadu, E.J., Ojarikre, I.H. and Njoseh, I. N. (2021). Numerical Solution of Fractional Integro-Differential Equation using Galerkin Method with Mamadu-Njoseh Polynomials. *Australian Journal of Basic and Applied Sciences*, 15(10): 13-19. DOI: 10.22587/ajbas.2021.15.10.2.

Meerschaert, M.M. and Tadjeran, C. (2004). Finite difference approximations for fractional advection-dispersion flow equations. *Journal of Computational and Applied Mathematics*,172(1):65–77.

Njoseh, I.N. and Mamadu, E.J. (2016a). Numerical Solutions of a Generalized Nth Order Boundary Value Problems using Power Series Approximation Method. *Applied Mathematics*, 7: 1215- 1224. http://dx.doi.org/10.4236/am.2016.711107

Njoseh, I.N. and Mamadu, E.J, (2016b). Variational Iteration Decomposition Method for the Numerical Solution of Integro-Differential Equations. *Nigerian Journal of Mathematics and Applications*, 25: 122-130

Njoseh, I.N. and Mamadu, E.J, (2017a). Solving Delay Differential Equations by Elzaki Transform Method. *Boson Journal Modern Physics*, 3(1): 214-219.

Njoseh, I.N. and Mamadu, E.J, (2017b). Variation Iteration Method for the approximate Solution of Nonlinear Burgers Equation. *Nigerian Journal of Basic and Applied Sciences*, 24(1): 70-75.

Ogeh K.O. and Njoseh, I.N. (2019). Modified Variational Iteration Method for solving Boundary Value Problems using Mamadu-Njoseh Polynomials. *International Journal of Engineering and Future Technology*, 16(4): 23-34.

Roop, J.P. (2006). Computational aspects of FEM approximation of fractional advection-dispersion equations on bounded domains in R^2, *The Journal* of *Computational* and *Applied Mathematics*, 193: 243-268.

Shen, S., Liu, F., Anh, V., Turner, I. and Chen, J. (2014). A novel numerical approximation for the space fractional advection-dispersion equation, *IMA Journal of Applied Mathematics*,79: 431–444.

Sweillam, N.H. and Khader, M.M. (2010). A Chebyshev pseudo-spectral method for solving fractional integro-differential equations. *ANZIAM Journal*, 51: 464-475.

Zhang, J. (2018). A stable explicitly solvable numerical method for the Riesz fractional advection–dispersion equations.*Applied Mathematics* and *Computation*, 332:209–227.

Zhang, X., Liu, L., Wu, Y., and Wiwatanapataphee, B. (2017). Nontrivial solutions for a fractional advection–dispersion equation in anomalous diffusion. *Applied Mathematics Letters*,66:1–8.