Inhibiting effects of Ito-type Brownian noise on the blow-up of solutions of nonlinear pantograph differential equation

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Accepted 29th November, 2021

This article studies the influence of multiplicative noise on the blow-up behavior of solutions of a nonlinear deterministic pantograph differential equation. The deterministic equation is perturbed by an Ito-type white noise and the unbounded growth rate is examined along Osgood condition. It is established that if the noise scaling parameter in the state dependent diffusion term is sufficiently strong, the presence of noise ensures that the blow up of solutions of the resulting nonlinear stochastic Pantograph delay differential equation is inhibited or prevented from occurring. However, the underlying deterministic differential equation, where noise is absent, will still admit solutions which blow up to infinity at finite time.

Key words: Inhibiting effects, blow-up, unbounded growth rate, nonlinear, stochastic pantograph differential equation, multiplicative Ito-type white noise, finite time.

INTRODUCTION

Ordinary differential equations (ODEs) and Delay differential equations (DDEs) usually arise in various areas of application, which include modeling of infectious diseases, physiological and pharmaceutical kinetics such as the body's reaction to CO_2 in circulating blood, navigational control of ships and aircrafts as well as fluctuation in prices of commodities. Amongst other works, there exists a collection of literature that indicate application areas of DDEs and ODEs and we cite the important works of Rihan et al. (2018) and Glass et al. (2021).

Although, at a particular period of time delay differential equations received more attention in the far eastern bloc (such as India, Pakistan, etc.) than in the west (such as the United States of America, the United Kingdom, etc.), we presently have a large volume of English language literature on the theory of DDEs and their numerical solutions. Various aspect treated and also associated with the asymptotic behavior of solution of DDEs include stability, oscillation, periodicity as well as explosion or blow-up of such solutions (Oksendel, 1992), Mao (1997), Jordan (2008) and Atonuje (2010).

An area of increasing interest regarding the growth and decay of solutions of DDEs is their asymptotic behavior of tending to infinity at some finite times. Many authors which include Baris and Wawioroko (2008), Ezzinbi and Jazor (2006), Goriety and Hiyde (1998) and Wiener (1993) had devoted a lot of effort in providing answers to questions regarding the growth of solutions of ODES and DDEs to infinity or their explosion times. Amongst such questions are: At what time does a solution of deterministic ODE or DDE blow up? Does the sequence of explosion times follow a given pattern? At what rate does a solution grow unboundedly to the blow-up time? What are the conditions for a solution of classical DDE to blow up?

A glance through the numerous literature on the theory of blow-up of unbounded solutions of deterministic DDEs indicates that the control or effect of Ito-type white noise in ensuring that the blow-up of solutions of deterministic ODEs and DDEs is prevented or suppressed is less studied. To the best knowledge of the authors, the first article which focused attention in this direction was the work of Atonuje and Testimi (2016), where the authors examined the contribution of noise in suppressing the blow-up of solutions of the Fokker- Planck equation.

In this present article, we employ the formalism of Atonuje and Tsetimi (2016) and present results which establish the role played by Ito-type white noise in preventing or suppressing the blow-up of solutions of a nonlinear classical Pantograph equation which possess the tendency of satisfying the Osgood condition.

PRELIMINARIES AND INSPIRATION FOR THE WORK

Exponential growth and decay of differential equations

Consider the differential equation

$$\frac{dx}{dt} = kx \tag{1}$$

Where t and x are variables, $k \neq 0$ is a constant. t is a measurement of time, x is a positive quantity over time, that is, x is a function of time. The number k is called the continuous growth rate if it is positive or continuous decay rate if it is negative. By separating variables, we have:

$$\frac{1}{k}\int \frac{dx}{x} = \int dt \tag{2}$$

$$\frac{1}{k}\ln x + c = t,$$

where c is a constant of integration

$\ln x = kt - kc$ (index form gives)

$$x = e^{kt-kc} = e^{-kc} \cdot e^{kt}$$

Here, e^{-kc} is a constant C. The solution is just

$$x(t) = Ce^{kt} \tag{3}$$

Equation 1 has infinitely many solutions for different values of C. Exponential growth occurs when $k > 0$ and exponential decay occurs when $k < 0$. For articles on unbounded growth and decay of solutions of ODEs and DDEs (Appleby et al., 2010; Appleby, 2005) (Figures 1 and 2).

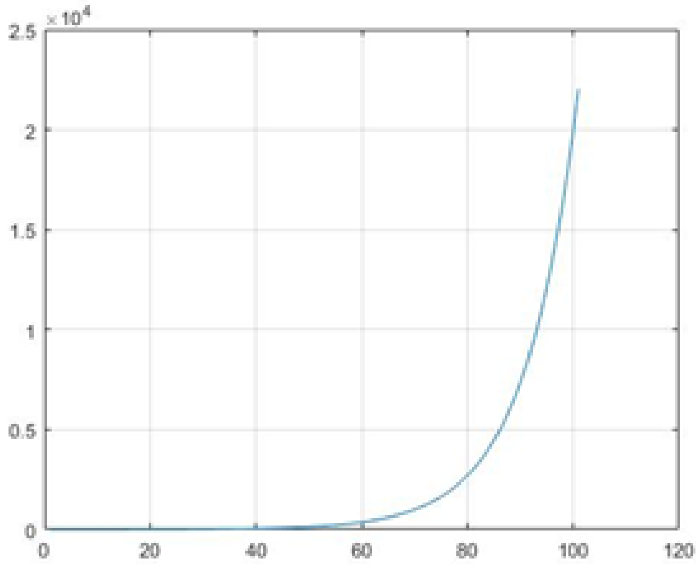

Figure 1. Graph of Equation 3 when k>0.

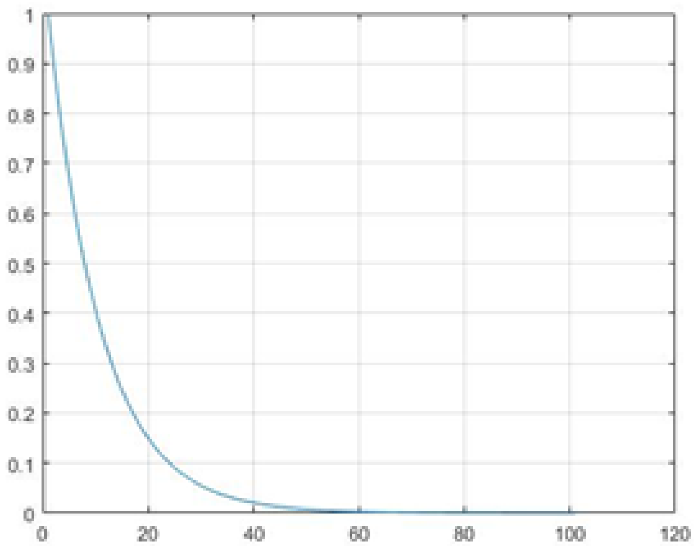

Figure 2. Graph of Equation 3 when k<0.

It is well known that solutions of ODDEs and ODEs are functions and as such they grow in their sample paths. Blow-up which is sometimes called explosion is a generalized concept that describes the fact that some solutions of dynamical systems grow unboundedly to infinity on the time horizon at some finite time if not checked. Consider the scalar deterministic ordinary differential equation

$$x^{'}(t) = x^2 \tag{4}$$

The solution of Equation 4, that is, $x(t) = \frac{x_0}{(1-x_0 t)}$ may become unbounded after a small lapse of time and may blow-up or explode to infinity at time $t = T(t_0) = \frac{1}{x_0}$, called its blow-up time.

In the present article, we study the blow-up behavior of a general functional differential equation of the nonlinear pantograph-type, which is given below:

$$\left.\begin{array}{c} x^{'}(t) = f\big(t, x(t), x(t-r), x(q_1 t), \dots, x(q_k t)\big) + g\big(x(t)\big), \quad t \geq 0 \\ x(t) = \psi(t), \ \ t \in [-r, 0] \end{array}\right\} \tag{5}$$

Where $f, g: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are analytic functions, $r > 0$ is a constant time lag or delay, $0 < q_1 < \cdots q_k < \cdots < 1, \ \ t > 0$ and $\psi \in C([-r, 0], \mathbb{R})$. The origin of pantograph equation is traceable to 1851 when a device called pantograph was used in the construction of electric locomotive. Since then pantograph equations have attracted the attention of Mathematicians and Engineers as a volume of research interest. They find application in electrodynamics, quantum mechanics, dynamical systems, number theory, population cell growth, astrophysics and analysis of overload current for electric locomotive, etc.

By solution of Equation 5 we mean a function $x(t)$ that is continuous on $[0, T]$ such that the derivative of $x(t)$ at each point $t \in [0, T]$ exists and also satisfies Equation 5 together with the initial condition. It is well known that the solution $x(t)$ of Equation 5 increases at a well specified rate given by

$$\lim_{t\to+\infty} \frac{x(t)}{t^{1/(1-\alpha)}} = (1-\alpha)\frac{1}{(1-\alpha)}, \qquad \alpha > 1$$

Definition 1
Suppose that there exists a unique continuous solution $x(t)$ of Equation 5 defined on some time horizon $[0, T]$ with $\alpha > 1$ such that

$$\lim_{t\to T^-} x(t) = +\infty, \tag{5}$$

then T is called the blow-up time of the solution and it is determined by the initial function ψ. Blow-up of solutions of dynamical systems provides useful application in detecting fatigue failure in steel and aluminum as well as other solid cutting tools. Blow-up time usually corresponds to time of ultimate damage in the particular solid material. It is also applicable in problems of shock waves, thermal explosion, thermal self-focusing beam structures in magneto-hydrodynamics, compression in gas dynamics and collapse of bridges. Such times are also applicable in crude oil refineries, where they are equivalent to the time of ultimate separation temperature of crude components.

Zhou et al. (2016) studied the uniqueness and explosion of solutions of a scalar nonlinear stochastic differential equation driven by an Ito-type Brownian motion of the form

$$\left.\begin{array}{l} dX = b\big(X(t)\big)dt + \sigma(X, \varepsilon)dB \\ X(0) = X_0 \end{array}\right\} \tag{6}$$

The authors proclaimed that for a non-negative and continuous $b(\cdot)$, the solution of Equation 6 blow-up in finite time if and only if

$$\int_0^\infty \frac{1}{b(s)} ds < \infty \tag{7}$$

This generalized condition is called the Osgood condition (Ince, 1994). Blow up of solutions od ODDEs and ODEs finds application in detecting fatigue failure in steel and aluminum as well as other solid cutting tools, problems of shock waves, thermal explosion, thermal self-focusing beam structures in magneto-hydrodynamics, compression in gas dynamics and collapse of bridges, crude oil refineries, etc.

In order to study the contribution or role of Ito-type white noise in ensuring that the blow-up of solution of Equation 5 is prevented, we shall stochastically perturb Equation 5 with a multiplicative white noise. The underlying stochastic Pantograph delay differential equation (SPDDE) is of the form:

$$\left.\begin{aligned} dX(t) &= \left[f\big(t, X(t), X(t-r), X(q_1 t), X(q_2 t), \ldots, X(q_k t)\big) + g\big(X(t)\big)\right]dt + \sigma h\big(X(t)\big)dW(t) \\ X(t) &= \psi(t), \ \ t \in [-r, 0] \end{aligned}\right\} \tag{8}$$

where σ is the noise scaling parameter, $h: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is an analytic function and the standard one-dimensional Wiener process $W(t)$ represents the Ito-type white noise. $W(t)$ has the following expectation and correlation properties: (i.) $E\big(W(t)W(s)\big) = t \wedge s$ where $t \wedge s = min(t, s)$, that is, $W(t)$ has orthogonality increments of non-overlapping intervals. (ii.) for $s, t, E(W(t)^2) = t$, $E[W(t) - W(s)] = 0$, $E\big(W(t) - W(s)\big)^2 = |t - s|$. (iii.) $E\{(W(t)/W_0^S) = W(s)\}$, $E\left\{\big(W(t) - W(s)\big)^2/W_0^S\right\} = t - s$. $\psi(t)$ is an $\mathbb{F}_{t_0}$-measurable $C([-r, 0], \mathbb{R})$-valued random variable such that $E|\psi|^2 < \infty$, $C([-r, 0], \mathbb{R})$ is a Banach space of all continuous paths from $[-r, 0] \to \mathbb{R}$, equipped with the supremum norm $\|\gamma\| = \text{Sup}_{s \in [-r,0]}|\gamma(s)|$, $\gamma \in \mathbb{C}$.

Definition 2

The solution of the SPDDE(8) is a stochastic process $\{X(t, \omega)\}_{t \geq 0}$ defined on a probability space $(\Omega, \mathbb{F}, P)$ consisting a space of elementary events Ω, a selected σ-algebra of events $\mathbb{F}$ in Ω and a measure P defined on $\mathbb{F}$ such that $P(\Omega) = 1$ is the probability of an almost sure event, where the measure P is has the following properties: Let S_n, $n = 1,2, \ldots \in \mathbb{F}$ then.

(i) $P(\emptyset) = 0$

(ii) If $S_k \cap S_r = \emptyset, k \neq r$, then $P(\bigcup_{k=1}^{\infty} S_k) = \sum_{k=1}^{\infty} P(S_k$

(iii) If $S_1 \subset S_2$, then $P(S_2 \backslash S_1) = P(S_2) - P(S_1)$

(iv) $P(\bar{S}) = 1 - P(S)$

(v) If $S_n \supset S_{n+1}, n = 1,2, \ldots$ then $P(\bigcap_{n=1}^{\infty} S_n) = \lim$

(vi) If $S_n \subset S_{n+1}, n = 1,2, \ldots$ then $P(\bigcup_{n=1}^{\infty} S_n) = \lim_{n \to \infty} P(S_n)$

The inhibiting effect of Ito-type noise on blow-up of solutions

Here, we provide results which establish that the solution of the deterministic Pantograph Equation 5 is monotonously increasing and satisfies the Osgood condition which makes it blow up or explode in finite time and also present results which explains the role of Ito-type white noise in preventing the blow up of solutions to be analyzed in the stochastic case.

Theorem 1

Assume that $f, g: \mathbb{R} \to \mathbb{R}$ satisfy the existence of $\bar{x} \geq 0$ and $f\big(x(t)\big) > 0, g\big(x(t)\big) > 0, \forall x > 0$ such that

$$f\big(t, x(t), x(t-r), x(q_1 t), \ldots, x(q_k t)\big) + g\big(x(t)\big) > 0, \quad \forall x \geq \bar{x}. \tag{9}$$

Then any solution of the deterministic PDDE (5) is monotone increasing in the upper endpoint of the interval [0,1) of existence for $\psi(0) > \bar{x}$ and blows up in finite time if $f\big(x(t)\big)$ satisfies the Osgood condition for blow-up of solutions of nonlinear deterministic ordinary differential equations, that is,

$$\int_x^{\infty} \frac{1}{f\big(x(t)\big)} dx < \infty, \quad \forall x > 0$$

Proof

We request that $[0, T)$ be taken as the maximum interval in which the solution $x(t)$ of Equation 5 exists and assume that $T > 1$ so that

$$x'(t) > 0 \quad for\ t \in [0,1) \tag{10}$$

$$x(1) \geq \psi(0) > \bar{x} \tag{11}$$

From the fact that $x > \bar{x}$ and Equation 9 we have that $x'(0) = f(\psi(0)) + g(\psi(0)) > 0$. Let $\tau = \inf\{t \in [0,1): x'(t) \le 0\} < 1$. One sees that $x(t)$ is increasing monotonically in the interval $[0,\tau] \subseteq [0,1)$ and as such, $x(\tau) \ge \psi(0)$. We can reverse the argument using Equation 9, that $x'(\tau) = f(x(\tau)) + g(x(0)) > 0$.

Hence, Equation 10 and 11 hold and by induction we see that $x(t)$ is monotone increasing in its maximum interval of existence and

$$x'(t) = f(t, x(t), x(t-r), x(q_1 t), \dots, x(q_k t)) + g(x(t)) > f(x(t))$$

which follows that the maximum interval of existence is bounded, since

$$< \int_{x(0)}^{x(t)} \frac{1}{f(x(t))} dx < \int_{x(0)}^{\infty} \frac{1}{f(x(t))} dx < \infty$$

That is $x(t)$ satisfies the Osgood condition and hence blows up in finite time.

The following result shows that the presence of the multiplicative Ito-type white noise in the stochastic Equation 8 ensures that the blow up of solution can never happen even if there is at most a linear growth in the drift term. We need the following assumptions:

B_1: $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfy the Uniform Lipschitz Condition
B_2: $r \in (0, \infty)$, $\sigma = 1$

B_3: The family of random variables $\{W(t), F_t, t \in T\}$ where $W(t)$ is F_t-measurable for all $t \in [0,T]$ is such that $E\{W(t)\} < \infty$ and $E\{W(t)/F_s\} \ge W(s)$, $s < t$, $s,t \in [0,T]$ that is, $\{W(t)\}_{t\ge0}$ is a semi-martingale for all $t \ge 0$ up to $\tau(x)$, the explosion time.

B_4: $\mathrm{Sup}_{x\in\mathbb{R}} \frac{xfx - \frac{1}{2}h(x)^2}{1+|x|^2} < \infty$

Theorem 2
Assume that B_4 holds and there exists a positive constant $L > 0$ such that the function h has at most a linear growth, in particular,

$$|h(x)| \le L(1 + |x|), \ \ x \in \mathbb{R}.$$

Assume also that B_1, B_2 and B_3 are satisfied, then the blow-up of the solution of the stochastic Pantograph Equation 8 can never happen.

Proof
Request that the general initial condition be bounded, i.e., for $X(0) = \psi(0)$, we allow the existence of value K_0 such that

$$\lambda k_0 = |X(0)| = |\psi(0)| \le K_0$$

Let

$$\tau_{n-1} = \inf\{t > \mu_{n-1}: |X(t)| = R\}$$

be a defined sequence of stopping times for some positive $R \ge K_0$, where

$$\mu_n = \inf\{t > \tau_n: |X(t)| = R + 1\}$$

Also let

$$\gamma_{k_0} = \inf\{t > 0: X(t) = K_0\}$$

For easy interpretation in Ito sense we write the stochastic Equation 8 in its integral form

$$X(t) = X(0) + \int_0^t [f(s, X(s), X(s-r), X(q_1 s), X(q_2 s), \dots, X(q_n s)) + g(X(s))] ds + \sigma \int_0^t h(X(s)) dB(s)$$

and as such

$$E|X|^2_{\infty,\tau_r\wedge\gamma_{k_0}} \leq 2E|X(0)|^2$$

$$+E\int_0^{\tau_n\wedge\gamma_{k_0}}[f(s,X(s),X(s-r),X(q_1s),X(q_2s),\dots,X(q_ns))+g(X(s))]ds+ME\int_0^{\tau_n\wedge\gamma_{k_0}}\{1+|X(s)|^2\}dB(s)$$

$$\leq 2E|X(0)|^2+E\int_0^{\tau_n\wedge\gamma_{k_0}}[f(s,X(s),X(s-r),X(q_1s),X(q_2s),\dots,X(q_ns))+g(X(s))]ds$$

$$+MT\int_0^T E|X|^2_{\infty,\tau_r\wedge\gamma_{k_0}}ds$$

Using the Gronwall's inequality and keeping in mind that assumption B_4 holds, one gets

$$E|X|^2_{\infty,\tau_r\wedge\gamma_{k_0}} \leq (2E|X(0)|^2+MT)e^{CT}$$

$$+\int_0^{\tau_n\wedge\gamma_{k_0}}[f(s,X(s),X(s-r),X(q_1s),X(q_2s),\dots,X(q_ns))+g(X(s))]ds$$

We now see that $K_0\to\infty$, hence, we have that $|X|_{\infty,\tau_r\wedge\gamma(X)}<\infty$ and so

$$\tau_r\wedge\gamma(X)<\gamma(X)$$

We see that $\gamma(X)=\infty$ follows from the fact that $\tau_r\to\infty$ as $T\to\infty$. As such, the function $f(t,X(t),X(t-r),X(q_1t),X(q_2t),\dots,X(q_nt))$ cannot satisfy the Osgood condition and as such, the solution of the stochastic Pantograph Eq.8 can never blow up.

CONCLUSION

We observe that assumption B_4 is the major condition which ensures that the Ito-type white noise is sufficiently strong in the state dependent diffusion term with respect to the drift function f and at that point, blow-up can never happen in Equation 8. However, the deterministic Equation 5 still admits a solution that blows up due to the absence of noise.

CONFLICT OF INTERESTS

The authors have not declared any conflict of interests.

REFERENCES

Appleby, J. A. D., McCarthy, M. and Rodkila, A. (2010). Exact growth rate of solutions of delay differential equations. *Proceedings of Neural Parrallel and Scientific Computations*. Vol.4, 37-42

Appleby, J. A. D. (2005). Decay and Growth Rates of Solutions of Scalar Stochastic Delay Differential Equations with Unbounded Delay and State Dependent Noise. *Stochastics and Dynamics*. 5(2).133-147. https://doi.org/10.1142/S0219493705001353 .

Atonuje, A. O., and Testimi J. (2016). Suppressive Influence of Time-Space White Noise on theExplosion of Solutions of Stochatsic Fokker-Planck Delay Differential Equations. *Journal of Mathematics and System Science.* Vol. 6, 284-290

Atonuje A. O. (2010). Analysis of Noise Generated Oscillation of Solutions of Stochastic Delay Differential Equations with Negative and Positive Coefficients. Jour. Inst. Of Mathematics and Computer Science (Mathematics Series), Vol. 23, No.3, 87-95

Baris, J., Wawiorko, E. (2008). Blow-Up Solutions of Cubic Differential Systems. *Journal of Mathematical Analysis and Applications.* 341(2), 1155-1162. https://doi.org/10.1016/j.jmaa.2007.11.020.

Ezzinbi, and Kad Jazar, M. (2006). Blow-up results for some non-linear delay differential equations. *Positivity, 10*, 329-341.

Rihan, F. A., Tunc, C., Saker, S. H., Lakshmanan, S. and Rakkiyappan, R (2018). Applications of Delay Differential Equations in Biological Systems. *Complexity*. Vol. 2018. https://doi.org/10.1155/2018/4584389

Glass, D. S., Jin, X. and Riedel-Kruse, I. H (2021). Nonlinear Delay Differential Equations and their Application to modeling Biological Network Motifs. *Nat Commun.* **12,** 1788. https://doi.10.1038/s41467-021-21700-8

Goriety, A. and Hiyde, C. (1998). Finite-time blow-up in dynamic systems. *Phy. Lett. A, 250*, 311-318.

Ince, E. L. (1994). Ordinary Differential Equations. Dover Publications, New York.

Jordan, P. M., (2008). A Note on Burgers Equation with Time Delay: Instability Via Finite Time Blow up. *Phy Lett. A.* 372 (42), 6363-6367.

Mao, X. (1997). Stochastic Differential equations and their Applications. Horwood Publishing Limited.

Oksendel, B. (1992). Stochastic Differential Equation: An Introduction With Application. Springer-verlag, Berlin.

Wiener, J., (1993). Generalized Solutions of Differential Equations. *World Scientific Singapore*.

Zhou, Y. C., Yang, Z. W., Zhang, H. Y., and Wang Y. (2016). Theoretical Analysis for Blow-Up Behaviours of Differential Equations with Piecewise Constant Arguments. *Journal of Applied Mathematics and Computation*. Vol 274 (2016) 353-361.