Numerical approach to the black-scholes model using Mamadu-Njoseh polynomials as basis functions

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This paper computed the option price using two different methods. The first method is to derive the analytical solution to the option price based on the classical Black-Scholes model. Next, the Black-Scholes differential equation was used to estimate the option price via a proposed numerical method called the reconstructed variational iteration method (RVIM) for stock price estimation. Here, the proposed method assumes the Mamadu-Njoseh polynomials as basis function in the estimation of the initial approximation via the orthogonal collocation method (OCM) to start the RVIM. Results were obtained using MAPLE and presented in tables and graphs for easy comprehension and interpretation.

Key words: Black-Scholes model, Mamadu-Njoseh polynomials, option pricing, European call option, orthogonal collocation method.

INTRODUCTION

The Black-Scholes model is one of the most significant amongst the many models in the backdrop of option pricing. The model itself was first proposed in 1973 by Fisher Black and Myron Scholes. However, it was the work of Robert Merton that provided the mathematical formulation of option pricing model; hence the term "Black-Scholes option pricing model (Black and Scholes, 1973; Merton, 1973). In the same year, Black-Scholes derive an equation via partial derivatives to analyze the relationship between stock price and time in option pricing. The equation is given below as:

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial t^2} - rC = 0 \qquad (1)$$

where, $C(S, t)$ denotes a function of stock price and time, S be a random variable representing the price of the stock, σ be the volatility rate of the stock return, r be the risk-free interest rate and t denotes the time in years.

The equation 1 can be described as a parabolic partial differential equation of the second order, which can be either solved analytically or numerically. Analytic methods for solving Equation 1 are complex, robust, and difficult to achieve. This is probably because many of these methods require perturbation, linearization, or some weak transformations. For instance, the Mellin transformation was adopted in (Jodar et al., 2005) to interpret the Black-Sholes model without due consideration of the dispersion condition. In like approach, the Black-Scholes model was reconstructed using discrete dividend (Company et al., 2006). The discrete dividend technique is a grouping of some generalized Dirac-Delta function and merged the Mellin transformation to obtain an integral formula. The solution is thus obtained numerically via numerical integration. The Black-Scholes model computes the theoretical call and European price options by ignoring dividend paid during option's expiration, which are summarized through the following assumptions (Shinde et al., 2012):

(i) No payment for dividends;
(ii) Upon expiration option must be exercised;
(iii) In the exchange no commission is paid;
(iv) Steady state interest rate;
(v) Consistent volatility rate;
(vi) No anticipation in the market circulation.

Let us now define the initial and boundary conditions for the European option. Now, the final time $t = T$ can be obtained from the concept of call option. For instance, if $t = T$ and $S > K$ (where K is the striking price), then the call option will be $(S-K)$. But, if $S < K$, then it will be worthless for a buyer to exercise the option. Thus, at $t = T$ the option price or value is called the final condition given as:

$$C(S, T) = max(S - K, 0). \qquad (2)$$

Let the value of $S = 0$ as $S \to \infty$. From equation (1) it is evident that the payoff is 0 iff $S = 0$. Thus, the boundary condition when $S = 0$ is

$$C(0, t) = 0, S \to \infty. \qquad (3)$$

This implies that options will be exercise and the call option value will be $(S - M)$. As $S \to K$, there is no impact on the call option value. Thus, the required boundary condition for the Black-Scholes Equation 1 is:

$$C(S, T) = S - Ke^{(-r(T - t))}, S \to \infty. \qquad (4)$$

In this paper, the Black-Scholes equation is considered numerically via a reconstructed variational iteration method (RVIM) employing the Mamadu- Njoseh polynomials as basis function.

OVERVIEW OF THE MODEL

The study considers the classical Black-Scholes model with single risky asset that follows a geometric Brownian motion:

$$dS_t = rS_t dt + \sigma S_t dW_t, \quad t \geq 0,$$

where ($W_t, t \geq 0$) is a standard Brownian motion, $\sigma \geq 0$ is the constant volatility, $r \geq 0$ is the constant risk-free rate and $S_0 \geq 0$ is the initial asset price.

Under these conditions, for any $t \geq 0$, the stock price S_t is given by the following formula:

$$S_t = S_0 e^{\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t}$$

We consider a security with time to maturity T and the payoff function:

$$P := (S, K) \to \begin{cases} 1 & S > K \\ 0 & S \leq K \end{cases} :$$

Payoff of the form

$$P(S) = \begin{cases} 1 & S > K \\ 0 & S \leq K \end{cases},$$

corresponds to a digital call options with strike price, K.

Several methods will be considered for computing the price of this security.

MATERIALS AND METHODS

Analytic solution via MAPLE 18

$$\_EnvStatisticsRandomVariableName := \Phi :$$

$$U := Statistics\text{:-}RandomVariable\left(LogNormal\left(\left(r - \frac{\sigma^2}{2}\right) \cdot T, \sigma \cdot \sqrt{T}\right)\right):$$

S_T can be represented in the form:

$$S_T := S_0 \cdot U :$$

where Φ is a lognormal random variable with parameters $T\left(r - \frac{\sigma^2}{2}\right)$ and $\sigma\sqrt{T}$.

The price of this option can be computed as the discounted expected payoff of the option:

$$e^{-rT}\mathbb{E}(P(S_T))$$

$$P\_analytic := P(S_T, K)$$

$$\begin{cases} 1 & K < S_0 \Phi \\ 0 & S_0 \Phi \leq K \end{cases}$$

$$V\_analytic := e^{(-r \cdot T)} \cdot Statistics\text{:-}ExpectedValue(P\_analytic) \text{ assuming } r > 0, \sigma > 0, S_0 > 0, K > 0, T > 0$$

$$e^{-rT}\left(-\frac{1}{2}\operatorname{erf}\left(\frac{1}{4}\frac{\sqrt{2}\left(\sigma^2 T - 2rT + 2\ln(K) - 2\ln(S_0)\right)}{\sigma\sqrt{T}}\right) + \frac{1}{2}\right)$$

The analytic result can be used to study the various market sensitivities. For example, we can symbolically compute the delta of our option as

$\Delta := diff(V\_analytic, S_0)$

$$\frac{1}{2}\frac{e^{-rT}e^{-\frac{1}{8}\frac{\left(\sigma^2 T-2rT+2\ln(K)-2\ln(S_0)\right)^2}{\sigma^2 T}}\sqrt{2}}{\sqrt{\pi}\,S_0\,\sigma\sqrt{T}}$$

Here is a formula for the Gamma:

$\mathbf{local}\,\Gamma := factor(diff(\Delta, S_0))$

$$-\frac{1}{4}\frac{e^{-rT}e^{-\frac{1}{8}\frac{\left(-\sigma^2 T+2rT+2\ln(S_0)-2\ln(K)\right)^2}{\sigma^2 T}}\sqrt{2}\left(\sigma^2 T+2rT+2\ln(S_0)-2\ln(K)\right)}{\sqrt{\pi}\,\sigma^3\,T^{3/2}\,S_0^2}$$

The symbolic formula can also be used to plot the option price as a function of the parameters.

Numerical approach

It is often difficult and complicated to obtain the analytical solution of most partial differential equations (PDEs), even if such analytical solution exist, it is presented in complex and robust forms. Thus, the need to solve PDEs numerically becomes essential. The variational iteration method (VIM) as proposed by He (1998) is one of the most famous iterative schemes available for the solution of most PDEs. The VIM is reconstructed using the Mamadu-Njoseh polynomials as Basis functions (Njoseh and Mamadu, 2016; Mamadu and Njoseh, 2016) to solve the Black-Scholes model numerically and efficiently.

Reconstructed variational iteration method

Given the Black-Scholes PDE (Andallah and Anwar, 2018) as (1), the correction functional is given as:

$$C_{m+1} = C_m + \int_0^t \lambda(\tau)\left(\frac{\partial C_m}{\partial \tau} + rS\frac{\partial C_m}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C_m}{\partial \tau^2} - rC_m\right)d\tau, m \geq 0, \quad (5)$$

where

$$\lambda(\tau) = (-1)^m \frac{(\tau - t)^{(m-1)}}{(m-1)!} \quad (6)$$

is called the general Lagrange multiplier (He, 1997; Abbasbandy and Shivanian, , 2009), m is the highest derivative of the PDE (1). Let

$$C_m = \sum_{i=0}^{N} a_i \varphi_i(t) \quad (7)$$

$a_i's$ are the unknown constants, and $\varphi_i(t), i = 0(1)N$, are the Mamadu-Njoseh polynomials constructed in the closed interval [-1,1] with respect to the weight function $w(x) = x^2 + 1$ (Njoseh and Musa, 2019; Ogeh and Njoseh, 2019; Mamadu and Ojarikre, 2019). The first five Mamadu-Njoseh polynomials as derived from the above properties are given below:

$$\varphi_0(x) = 1,\ \varphi_1(x) = x,\ \varphi_2(x) = \frac{1}{3}(5x^2 - 2),\ \varphi_3(x) = \frac{1}{5}(14x^3 - 9x)$$

and

$$\varphi_4(x) = \frac{1}{648}(333 - 2898x^2 + 3213x^4)$$

The study then proceeds to determine the initial approximation via the orthogonal collocation method (OCM). Using the initial condition from the Black-Scholes equation and (7) at $N = 3$, we have

$$\sum_{i=0}^{N} a_i \varphi_i(t) = C(S, 0) \quad (8)$$

Collocating Equation 8 at the roots of $\varphi_4(x)$, we have the matrix equation

$$Ax = b, \quad (9)$$

where

$$A = \begin{pmatrix} 1 & 0.3676425560 & -0.4413982517 & -0.5226219310 \\ 1 & -0.3676425560 & -0.4413982517 & 0.5226219310 \\ 1 & 0.8756710201 & 0.6113328923 & 0.303892220 \\ 1 & -0.8756710201 & 0.6113328923 & 0.303892220 \end{pmatrix}, \quad x = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}$$ and

$$b = \left(C_j(S,0)\right)^T, j = 0,1,2,3$$

Solving Equation 9 yields the initial approximation to the iterative scheme 5. The algorithm below is useful for executing the proposed method RVIM via MAPLE 18 for the Black-Scholes model.

ALGORITHM: RVIM FOR SOLVING BLACK-SCHOLES MODEL

Black-Scholes Model (C_0 ,N, λ,∈, S, K, r, σ, T).
This algorithm computes approximate solution of

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial t^2} - rC = 0$$

given an initial approximation C_0. Here, the function $C(S,t)$ is continuous and has a continuous derivative.
INPUT: Approximate solution C_m+1 ($m \leq N$) or message of failure.
For $m = 0, 1, 2, \cdots, N - 1$ do:

1. Compute λ

2. if $\lambda = 0$ then OUTPUT 'Failure'. Stop (procedure complete successfully)

3. Else compute

$$C_{m+1} = C_m + \int_0^t \lambda(\tau)\left(\frac{\partial C_m}{\partial \tau} + rS\frac{\partial C_m}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_m}{\partial \tau^2} - rC_m\right)d\tau, m \geq 0$$

(procedure complete successfully)

4. $|C_{m+1} - C_m| \leq \in |C_m|$ then OUTPUT C_{m+1} stop.

(procedure complete successfully)
End.

5. OUTPUT 'Failure'. Stop.

RESULTS

Let us have an assumed call option value with strike price $K = 100$, risk free interest rate $r = 0.10$, $T = 1.0$ (time to expiration), volatility, $\sigma = 0.10$. The call option value is plotted over the stock price $70 \leq S \leq 130$ for the analytic method and the RVIM (using the third approximation) as shown in Figures 1 and 2. Results are also presented in Table 1 for comparison between the analytic method and RVIM.

DISCUSSION

The study successively implemented both the analytic method and RVIM for the Black-Schole model resulting to some fascinating numerical evidences presented in Table 1, and in Figures 1 and 2, respectively. For a European call option with $K = 100$, $r = 0.10$, $T = 1$, σ = 0.10 and $70 \leq S \leq 130$, it was observed that RVIM converges rapidly to the analytic solution with a maximum error of order 10^{-7}. The results obtained in this are better than those available in

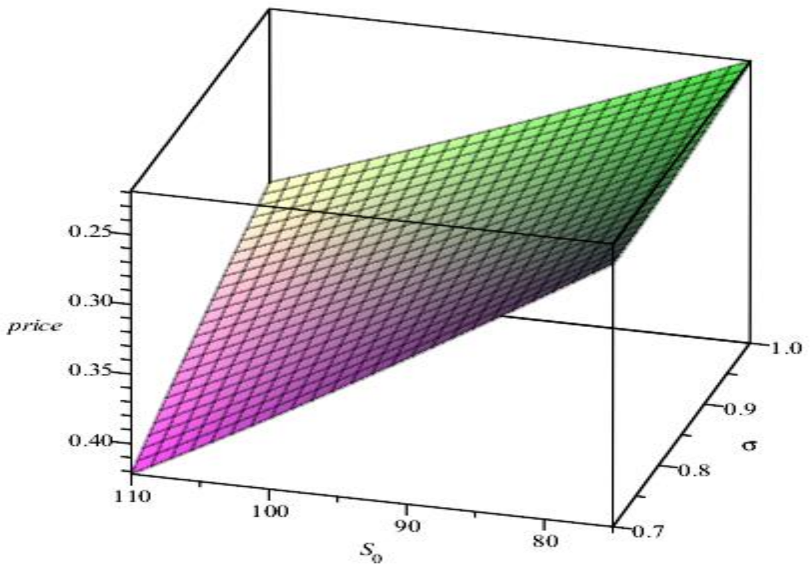

Figure 1. Analytic option price as a function of the parameters.

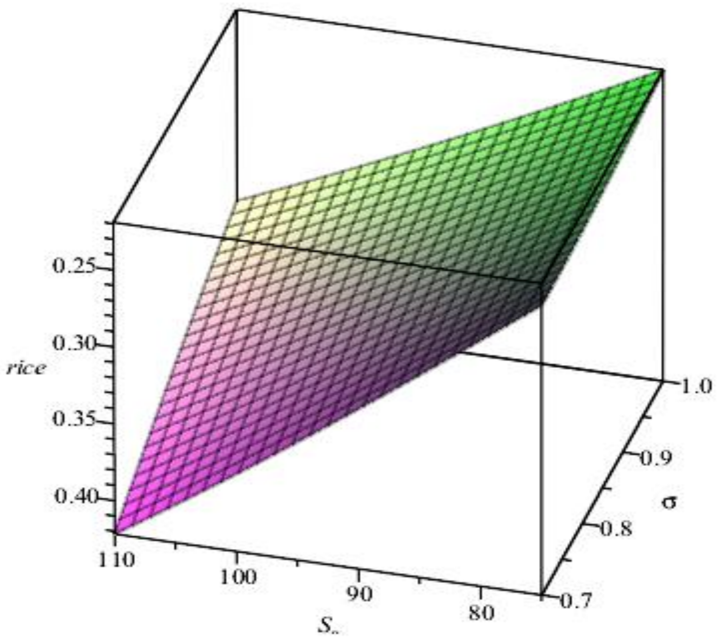

Figure 2. Plot of RVIM for the Black-Schole model.

Table 1. Comparison of results between the Analytic and RVIM solutions.

x	Analytic	RVIM	Error
0.02	18.9052	18.9052	2.10E-07
0.04	19.1188	19.1188	1.64E-06
0.06	19.3329	19.3329	5.53E-06
0.08	19.5476	19.5476	1.31E-05
0.1	19.7628	19.7627	2.56E-05
0.12	19.9784	19.9784	4.43E-05
0.14	20.1947	20.1946	7.04E-05
0.16	20.4114	20.4113	1.05E-04
0.18	20.6286	20.6285	1.50E-04
0.2	20.8464	20.8462	2.06E-04

the literature (Andallah and Anwar, 2018) which has a maximum error of order 10^{-6}. Also, the estimated approximate solution for stock price is 0.4084046091.

Conclusion

This paper has considered the numerical approach of the Black-Scholes model applying the Mamadu-Njoseh polynomials as basis function. The model has been studied and solved both analytically and numerically. The study has formulated the analytic approach via MAPLE 18 procedure for the Black-Schloes equation. The RVIM was formulated via a well-poised algorithm for the Black-Scholes equation. The resulting numerical evidences as presented above are in agreement with the qualitative behavior of the Black-Scholes equation. Also, comparison with the finite difference method (Andallah and Anwar, 2018) shows that the RVIM is a better numerical solver of the Black-Scholes equation.

CONFLICT OF INTERESTS

The author has not declared any conflict of interests.

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