Analysis of the extreme kinematic behavior that describes marine wave group velocity

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This study concerns the kinematic manifestations exhibited among marine parameters. This is more identified when the sea body varies slowly. Wave group speed is described in this study due to its obvious relation with wave group energy fluxes. Wave group speed is one of the parameters of ocean. On a slowly varying ocean, the evolution of wave group velocity is governed by the kinematic quasi-linear differential equation; thus, it follows the characteristic lines with their inherent inter- crossing, which may lead to the development of shock waves in the process. In the event of sea waves with large amplitudes, extreme growth that describes wave group velocity increases the energy fluxes in the process. The behavior of the group wave velocity established in this analysis modifies the wave amplitude growth but also increases the wave energy fluxes significantly. The situation may increase the destructive tendency which is usually observed. This is likely followed by the appearance of the induced large amplitude sea wave train and associated enhanced energy fluxes.

Key words: Kinematic, quasi-linear, energy fluxes, complex patter

INTRODUCTION

Kinematic theory is widely involved in the study of marine physical phenomena. These include those that can be classified as highly complex in their manifestations. These are analyzed effectively by employing the methods of characteristics developed by Light hill and Whitham (1955). The important ideas provided by this method may explain some important, physical manifestations associated with hyperbolic shockwave events with all their generalizations. The remarkable fact is that both progressive distortion of wave profile and related development of discontinuity in wave profile are typical non -linear developments that are product of wave kinematic behaviors.

It has been established (Ifediora, 2015) that ocean wave parameters are kinematic in behavior through their physical manifestations. Some of these parameters are wave frequency, wave number, and wave phase and group velocity, respectively. This is always the case in the ocean body that is slowly varying in time and space (Okeke, 2020). Group wave velocity is essentially among the leading factors in the consideration of the energy fluxes in wave group propagations. Consequently, an interesting and essential property is introduced in this study. This is the effect of wave group velocity that is a kinematic parameter and at the same time, the propagation speed of a wave group energy fluxes. Discussions on this will follow subsequently.

Velocity V(x, t) associated with the dominant wave group

The wave process is described in (x, t) coordinate system. Accordingly, the wave front is moving such that x-axis is perpendicular to it; t is the time duration. Take the successive crest elevation as $\eta_1(x,t) and\, \eta_2(x,t)$. The total momentum, M, between the elevations per unit volume is,

$$M = \rho \int_{\eta_2}^{\eta_1} V(x,t)dx \tag{1}$$

$\rho = constant\ density$

The rate of change M with t (using Leibnitz rule) provides that.

$$\frac{d}{dt}\int_{\eta_2}^{\eta_1} V(x,t)dx = \int_{\eta_2}^{\eta_1} \frac{\partial V}{\partial t}dx + \eta_{1t}V(\eta,t) - \eta_{2t}V(\eta_2,t) = 0$$

$$\eta_1 \to \eta_2\ , \quad \frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} = 0, \tag{2a}$$

As The development appears to suggest that dominant group velocity can be described in the context of kinematic consideration (Whitham, 2002). Alternatively, use total derivative to obtain,

$$\frac{dV}{dt} = V_t + V\,V_x = 0 \tag{2b}$$

Thus, along the characteristic line joining the point $(\xi,0)and\ (x,t),(Stoker, 1957)$.

$V(x,t) = V(\xi,0) = V(\xi) = constant.$

Thus, $V(\xi)$ is constant along the straight line joining the points $(\xi,0)\ and\ (x,t)$ being a characteristic line. The characteristic curve for equation (2a) is as follows:

$$dx = V(\xi)dt, x = tV(\xi) + A_0,$$

With $V(x,0) = f(x)$ as initial Cauchy data. When t = 0, x = $\xi \therefore A_0 = \xi$ and thus,

$$x = \xi + tV(\xi). \tag{3}$$

$$V(x,t) = f(\xi) \tag{4}$$

Equations 3 and 4 represent the equations of the characteristics for Equation 2.

From Equation 3, we have:

$$1 = \xi_x + tV^1(\xi)\xi_x = \xi_x\big(1 + tV(\xi)\big), \tag{5}$$

and

$$0 = \xi_t + V(\xi) + tV^1(\xi)\xi_t = V(\xi) + \xi_t(1 + tV^1)$$

$$\xi_t = \frac{-V(\xi)}{1 + tV^1(\xi)}, \quad \xi_t = \frac{1}{1 + tV(\xi)}. \tag{6}$$

Equation 4 provides the following:

(i) $$V_x = f^1(\xi)\xi_x = \frac{f^1(\xi)}{1 + tV^1(\xi)}, \tag{7}$$

(ii) $$V_t = f^1(\xi)\xi_t = \frac{-f^1(\xi)V(\xi)}{1 + tV^1(\xi)}. \tag{8}$$

If $V^1(\xi)$ is negative, $t = -\frac{1}{V^1(\xi)}$, suggesting the time for the commencement of singular behavior that describes the solution of the kinematic Equation 2a.

Intersection of characteristic lines

Consider two characteristics originating from two points on x – axis namely $(\xi,0)$ and $(\zeta,0)$. The equations are: $(\zeta > \xi)$

$$x = \xi + tV(\xi), x = \zeta + tV(\zeta) \tag{9a,b}$$

$\zeta > \xi\ and\ V(\xi\ ) > V\ (\zeta)$

Thus, $\frac{1}{V(\xi)} < \frac{1}{V(\zeta)}$; hence, the two lines will intersect and this gives rise to the singular solution at $t = t_0$, where

$$t_0 = \frac{-(\zeta - \xi)}{V(\zeta) - V(\xi)} \tag{10}$$

Since $V(\zeta,) < V(\xi), t_0 > 0$

Equation 10 provides the time $t = t_0$ for the on-set of shockwave manifestation in the quasi-linear kinematic process. The physical reality is that the picture is not as simple as it appears. Instead, we have a family of characteristic lines intersecting and inter- crossing among themselves. The development may give rise to extreme large amplitude wave train. It is important to mention this fact in marine physics. In the framework of the frequency modulated wave train evolution, characteristic lines and rays are identical in their physical representations and in water wave theory.

Various forms of initial data

(i) $$f(x) = x, 0 < x < \infty,\ t = 0 \tag{11}$$

The solution of Equation 2 is of the form in this case through Equations 3 and 4

$V(\zeta) = \xi, \quad x = \xi + t\xi, \quad for\ which,$

$$\xi = \frac{x}{1+t} \qquad (12a)$$

$$V(x,t) = \frac{x}{1+t} \qquad (12b)$$

Even with the simple linear initial data (11), $V(x,t)$ is still explosive when $t = -1$, however t> 0, in Equation 11, hence $V(x,t)$ is smooth.

(ii) $f(x) = \cos x, 0 < x < \infty,$

In this case, the solution of Equation 2 is of the form,

$$V(\xi) \qquad (13)$$

$$x = \xi + t\cos\xi. \qquad (14)$$

We need to solve for ξ in Equation 14.
To obtain complete analytical solution for Equation 14, take

$$\cos\xi = 1 - \frac{\xi^2}{2} + 0\ (\xi^4).$$

Therefore Equation 13 gives:

$$V(\xi) = 1 - \frac{\xi^2}{2} \qquad (15)$$

Equation 14 provides:

$$x = \xi + t\left(1 - \frac{\xi^2}{2}\right)$$

That is ,

$$t\xi^2 - 2\xi + 2(x-t) = 0 \qquad (16)$$

$$\xi = \frac{2 \pm [4 - 4t(x-t)]}{2t}$$

$$\frac{1}{t}\left[1 \pm (1 - t(x-t))^{1/2}\right] = \frac{1}{t}[1 \pm (1 - tx + t^2)^{1/2}]$$

But,ξ is real value of x. This condition implies that $1 + t^2 \geq tx$.
Subsequent calculations are based on this assumption.

$$\xi^2 = \frac{1}{t^2}\{[2 - t(x-t)] + 2(1 - t(x-t)^2)^{1/2}\} \qquad (17)$$

$$V(x,t) = 1 - \frac{\xi^2}{2} \qquad (18)$$

Figure 2 appears to suggest an unexpected growth of an initial pulse, modeled with first two terms of cosine series expansion. This is exceptionally high for the non – dimensional time t = 5 if 12 s wave dominant group is considered as it appears in Figure 2.

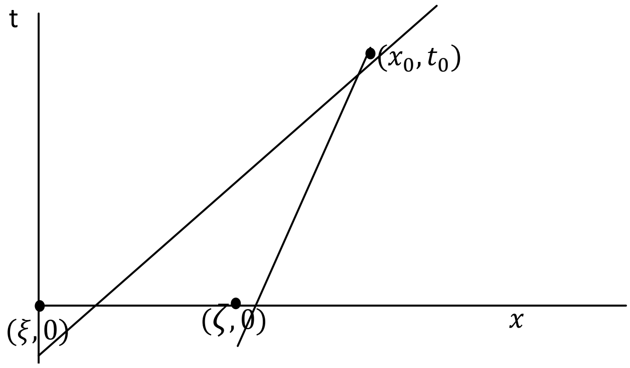

Figure 1. Inter-crossing of two characteristic lines or rays.

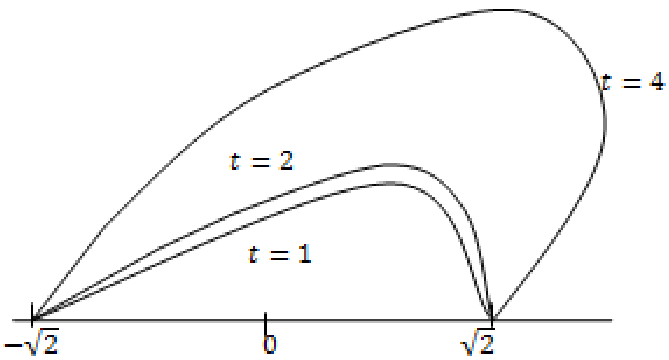

Figure 2. Propagation of initial pulse.

$$= \cos\xi.$$

The parameterized approach

Consider the case in which the physical process described by the Equation 2 is induced by a unit source. Thus,

$$V_t + VV_x = 1. \qquad (19a)$$

Choose σ and $\tau (\neq t)$ such that when $\tau = 0$
$x = V = \sigma, \quad t = 2\sigma,$ and

$$\frac{dt}{1} = \frac{dx}{V} = \frac{dV}{1} = d\tau. \qquad (19b)$$

From Equation 19a, and b is the equation of characteristics for (19a).

Thus, $dV = d\tau, V = \tau + A$ $when$ $\tau = 0, A = \sigma$, for which

$$V = \sigma + \tau . \quad (19c)$$

dt $= d\tau$, t $= \tau + B$ when $\tau = 0$, B = 2σ, for which

$$t = 2\sigma + \tau . \quad (20)$$

dx = $Vd\tau$ = (σ + τ) $d\tau$.

$$x = \tau\sigma + \frac{1}{2}\tau^2 + B, when\ \tau = 0, x = \sigma$$

Then B = σ and

$$x = \frac{1}{2}\tau^2 + \tau\sigma + \sigma . \quad (21)$$

The exercise now is to determine σ and τ using Equation 20 and 21 as functions of x and t, hence,

$V = V(x,t)$, using Equation 19c

From Equation 20 and 21, $t = \tau$ + 2σ, $\tau = t$ - 2σ.

Thus,

$$2(x - \sigma) = (t - \sigma)^2 - \sigma^2. \quad (22)$$

Again, from Equation 20 and 21,

$$2x - t = \tau(t - 1) . \quad (23)$$

Thus,

$$\tau = \frac{2x-t}{t-1}. \quad (24)$$

From Equation 22

$$2x - 2\sigma = t^2 - 2t\sigma, 2x - t^2 = 2\sigma(1 - t).$$

$$\sigma = \frac{1}{2}\frac{(t^2 - 2x)}{(t - 1)} . \quad (25)$$

From Equation 19b.

Finally ,

$$`V(x,t) = \tau + \sigma = \frac{2x-t}{t-1} + \frac{t^2-2x}{2(t-1)}.$$

$$i.e\ V(x,t) = \frac{1}{2(t-1)}[2x + t^2 - 2t] . \quad (26)$$

Figure 3 suggests the profile of $V(x,t)$. The apparent explosive behavior of the characteristics describing an identical kinematic process illustrates the singular tendency introduced by the factor $t - 1$ in the denominator of Equation 26 and is clear in Figure 3. Furthermore, as $t \to \infty$ or $x \to \infty$, $V(x,t) \to \infty$. Thus, the process has no finite upper bound suggesting an eventual explosive solution. This can give rise to large amplitude shock phenomena which may be of physical significance (Petrova, 2007; Polnikov, 2008) in marine physics.

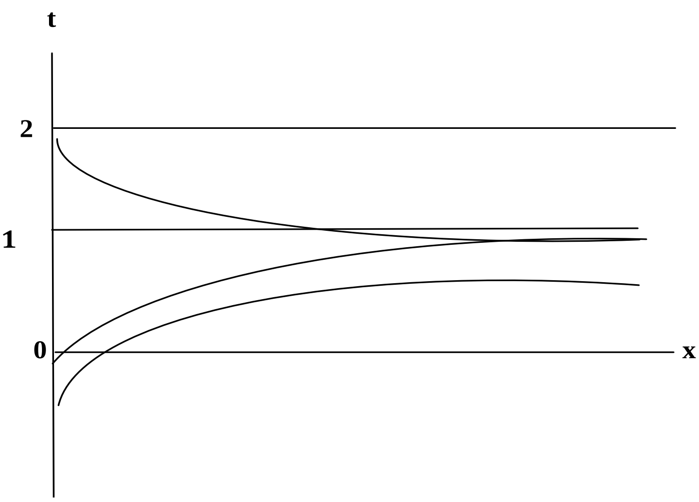

Figure 3. The Profile of $V(x,t)$ in the space- time coordinate system.

Modulating wave train with high crest elevation

We discuss the critical, though interesting, role associated with group velocity $V(x,t)$ in the development of sea wave train with extreme high crest elevation. In this consideration, we shall employ the above discussion to describe some of the dynamical features in these marine processes. These processes include those that exhibit extreme behavior in their physical manifestations. The wave group velocity $V(x,t)$ in this consideration plays a dominant role in water wave dynamics. At this starting point, we mention the role associated with $V(x,t)$ very briefly in the following considerations. Then, we incorporate this in our conclusion. For details, see for example, Kundu (1990) and Stokes (1957). The first order wave field potential $\varphi(x,z,t)$ is described by the fluid representation for irrotational and un compressible fluid (Stoker, 1957):

$$\varphi(x,z,t) = \eta_0\,(\omega/g)\frac{\cosh k\,(z+h)}{\sin kh}\cos(kx-\omega t). \tag{27}$$

η_0 = wave amplitude or crest elevation, z is the vertical coordinate normal to x- axis, h = the water depth measured from undisturbed sea surface, ω = wavedominant frequency with wave number k, g is the acceleration due to gravity. The linear form of Bernoulli's equation for the linear pressure wave $P(x,z,t)$ provides,

$$P(x,z,t) = -\rho\frac{\partial\varphi}{\partial x}. \tag{28}$$

We have neglected the nonlinear term and atmospheric forcing term in equation (28). The energy flow ϵ per unit area in the fluid is provided by,

$$\epsilon\frac{\partial\varphi}{\partial \mathrm{t}}. \tag{29}$$

$\frac{\partial\varphi}{\partial x}$ = horizontal fluid particle velocity component

The total energy flow through the vertical –h < z < 0 and of unit width is given by:

$$E\int_{-h}^{0}\varphi_x\varphi_t\;dz\;. \tag{30a}$$

From Equations 27 and 30

$$E = -\left(\eta_0\frac{\omega}{g}\right)^2 k\omega\frac{\sin^2(kx-\omega t)}{\sin h^2 kh}\int_{-h}^{0}cosh^2k\,(z+h)dz\,. \tag{30b}$$

But

$$V = \frac{c}{2}\left(1+\frac{2kh}{\sin h\,zkh}\right). \tag{30c}$$

Where c is the phase speed associated with component wave mode,

$\overline{\sin^2(kx-\omega t)} = \frac{1}{2}$.

The bar indicates the statistical mean over a wave length or period $\left(\frac{2\pi}{k}\;or\;\frac{2\pi}{\omega}\right)$.

From Equation 30b) using Equation 30c

$$\bar{E} = \frac{\rho g\eta_0^2}{2}V(x,t)\,. \tag{31}$$

Equation 31 is the expression for the average energy transmission in a sinusoidal wave group.

Thus, $\bar{E} = eV$. (32)

$e = potential + kinetic\ energy$

Identical to the mass conservation principle, the energy conservation equation (Kundu, 1999) is,

$$\frac{\partial\bar{E}}{\partial t} = \frac{\partial}{\partial x}(VE). \quad (3$$

$\eta_0 = \frac{A}{2}$, where A = wave height from the crest height to the trough. Thus, the expression for wave energy is given by: $e = \frac{1}{4}\rho g A^2$. It follows that e is proportional to the square of wave amplitude. Equation (33) is now of the form :

$$\frac{\partial A^2}{\partial t} = \frac{\partial}{\partial x}(VA^2)\,. \tag{34}$$

In the above, the x-axis is usually horizontal but rotated perpendicular to the wave vector i.e. perpendicular to the wave front elevation. Following Brown (2001), Equation 34 has representation,

$$V\,\Lambda\,A^2 = constant. \tag{35}$$

Where Λ is the separation factor which denotes the distance between two neighboring characteristic lines (rays). At the inter-crossing of these lines, Λ→ 0 and A →∞, from Equation 35. There are large numbers of such lines that are not parallel and thus, are involved in the process of inter-crossing and hence form a complex pattern (caustics and focuses). The scenario may lead to sequence of events in the form of large amplitude ocean wave which is the important physical manifestation.

CONCLUSIONS

Very interestingly, this study introduces a more complex factor. At the inter-crossing and when the time $t = -\frac{1}{V^1(\xi)}$, in Equation 8 or 9, the energy fluxes (flow) become extremely high following the behavior of group velocity V(x, t) which is the propagation speed of the energy flow in Equations 33 and 34. Since there is a limiting constant in Equation 34, A may not likely grow as expected.

In totality, the resulting wave height will be moderately extreme but energy fluxes so generated may be unusually great following the earlier formulation, concerning the related group velocity kinematic behavior in this study. Thus, this analysis may have extended the knowledge emanating from the previous findings significantly (Petrova, 2005; Brown, 2001; Smith, 2002).

CONFLICT OF INTERESTS

The authors have not declared any conflict of interests.

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