Existence of solutions and stability analysis for a fractional helminth transmission model within the framework of Mittag-Leffler kernel

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In recent years, the many tools from fractional calculus have been extensively used in the mathematical modeling of infectious diseases. In this paper, an integer order helminth transmission model proposed by Lambura et al. is extended to a fractional model by incorporating the fractional Atangana-Baleanu-Caputo derivative. Certain basic features such as non-negativity of solutions, invariant region within which the model equations are epidemiologically meaningful as well as equilibrium points and basic reproduction number are explored. Furthermore, the existence, uniqueness and Ulam-Hyers of the associated fractional model are explored via a fixed point technique and generalized Gronwall inequality.

Key words: Helminth disease, Atangana-Baleanu, fixed point technique, Ulam-Hyers stability, generalized Ulam-Hyers stability.

INTRODUCTION

The adverse effect diseases caused by parasitic worms cannot be underestimated. A major variant of existing parasitic worm diseases is the soil-transmitted helminthiasis which is caused by intestinal parasitic nematode such as hookworm species (*Necator americanus* and *Ancylostoma duodenale*), *Ascaries lumbriocoides* (roundworm) and *Trichuris trichiura* (whipworm) (Jennifer and Jurg, 2008; Lambura et al., 2020). People get infected with helminthiasis by ingesting unwashed/undercooked vegetables, unpeeled fruits or water already contaminated by parasitic eggs. Soil transmitted helminths are one of the most prevalent Neglected Tropical Diseases (NTDs) affecting the poorest and resource-constrained populations especially in Tropical Regions of the world (Pullan et al., 2014). Available records show that over 1.5 billion people are infected with helminthes throughout the world (Michael et al. 2006). In many cases, this parasitic infection leads to high morbidity and severe pathological complications such as organ failure (Codella et al., 2015). More severe complications arise in children of school age as they are the most infected (Truscott et al., 2016). The intensity of helminth disease infection can lead to social, economic and educational deficiency among children. Hence, effective and adequate control measures are necessary. A possible way of eliminating helminth infection is by reducing the concentration of worms/parasites in a host. However, this cannot be achieved without reducing the population density of the parasite from contaminated environment. Current control strategy consists of preventive chemotherapy (PC) which is aimed at school age children and pre-school children (WHO, 2012).

Over the years, mathematical models have been employed to improve our understanding of disease dynamics as well as to provide tools for assessing and evaluating effective control measures. The mathematical model for the helminth infection can be traced to the works of

Anderson and May (1978; 1982a; 1982b). Lambura et al. (2020) studied a deterministic compartmentalized mathematical model for soil-transmitted helminth disease with optimal controls. Their model incorporates integer order ordinary derivatives. They obtained basic reproduction number of the model without control and showed that both the disease-free and endemic equilibrium points are asymptotically stable under given threshold conditions. Furthermore, using incremental cost-effective ratio, they presented the cost effectiveness of the control measures and their results showed that the combination of health education and sanitation is the best strategy to combat the helminth infection.

Mathematical models with integer-order derivatives do not adequately account for hereditary and memory effects associated with most real life processes. Thus, advances in the field of Fractional Calculus have helped mathematicians to develop models with fractional (or arbitrary or non-integer) order differential or integral operators. In recent times, mathematical models with fractional order derivatives have become a central area of research studies as they efficiently incorporate the evolution-related realities and evidences inherent in the systems they model. This has led researchers to extend the notions of classical calculus to fractional calculus and incorporate these notions into models rising from mathematical biology. Various notions of fractional differential operators have been investigated over the years. Among these, the Riemann-Liouville and Caouto fractional derivatives (Kilbas et al., 2006) are the most widely studied in existing literature. The Caputo derivative has some disadvantages which include the singularity property associated with power-type kernel function. Recently, Caputo and Fabrizo (Caputo and Fabrizio, 2015) proposed the so-called Caputo-Fabrizo fractional derivative whose kernel is the exponential function. However, this derivative has limitation due to the locality of the kernel. To overcome this limitation, Atangana and Baleanu (Atangana and Baleanu, 2016) proposed the so called Atangana-Baleanu fractional derivative which incorporates the Mittag-Leffler function as a non-singular and non-local kernel.

In this paper, the integer order helminth transmission model proposed by Lambura et al. (2020) is extended by incorporating the fractional order derivative in the Atangana-Baleanu sense. To this end, we first recall some very important basic model properties associated with the normalized version of the integer order model. Next, we explore the existence and uniqueness of solutions to the fractional order model via a fixed point argument. Finally, we demonstrate that under certain conditions the stability of the fractional model in the sense of Ulam-Hyers is ensured.

PRELIMINARIES

In this section, we present some notions and preliminary concepts regarding fractional differential and integral operators that will be used in this work. In the sequel, we denote by $\Gamma(\cdot)$ the gamma function.

Definition 1 *Let* $\phi \in H^1(a,b)$, $a < b$ *and* $0 < \sigma \leq 1$. The Atangana–Baleanu-Caputo (ABC) fractional derivative defined in Atangana and Baleanu (2016) is given by,

$$ {}^{ABC}D_t^{\sigma}\phi(t) = \frac{\mathbb{B}(\sigma)}{1-\sigma}\int_0^t \phi'(\omega)\, E_\sigma\left(-\frac{\sigma}{1-\sigma}(t-\omega)^\sigma\right)d\omega, \quad t > 0, \tag{1} $$

where $\mathbb{B}(\sigma)$ is the normalization function which satisfies the property: $\mathbb{B}(0) = \mathbb{B}(1) = 1$ and $E_\sigma(\cdot)$ denotes the one-parameter Mittag-Leffler function (Podlubny, 1999) defined by,

$$ E_\sigma(z) = \sum_{i=0}^{\infty}\frac{z^k}{\Gamma(\sigma k+1)}, \quad \mathrm{Re}(\sigma) > 0, z \in \mathbb{C}. \tag{2} $$

Definition 2 The fractional integral (Atangana and Baleanu, 2016) associated with the ABC derivatives is defined by,

$$ {}^{ABC}I_t^{\sigma}\phi(t) = \frac{1-\sigma}{\mathbb{B}(\sigma)}\phi(t) + \frac{\sigma}{\mathbb{B}(\sigma)\Gamma(\sigma)}\int_0^t \phi(\omega)\,(t-\omega)^{\sigma-1}d\omega. \tag{3} $$

Equivalently, it is easy to see that the integral in (3) can be rewritten as,

$$ {}^{ABC}I_t^{\sigma}\phi(t) = \frac{1-\sigma}{\mathbb{B}(\sigma)}\phi(t) + \frac{\sigma}{\mathbb{B}(\sigma)}(I_t^{\sigma}\phi)(t), $$

where $I_t^{\sigma}\phi$ denotes the Riemanna-Liouville fractional integral operator (Samko et al., 1993; Kilbas et al., 2016).

Definition 3 The Laplace transform corresponding to the ABC fractional derivative (Atangana and Baleanu, 2016) for a function $\phi(t)$ is defined by:

$$ \mathcal{L}[{}^{ABC}D_t^{\sigma}\phi(t)] = \frac{\mathbb{B}(\sigma)}{s^{\sigma}(1-\sigma)+\sigma}[s^{\sigma}\mathcal{L}[\sigma(t)] - s^{\sigma-1}\phi(0)]. \tag{4} $$

Lemma 1 *Let $\sigma \in (0,1]$.* Then the solution of the time-fractional initial value problem,

$$ \begin{cases} {}^{ABC}D_t^{\sigma}\phi(t) = h(t), \\ \phi(0) = \phi_0 \end{cases} \tag{5} $$

is given by the integral equation (Atangana and Baleanu, 2016),

$$ \phi(t) = \phi_0 + \frac{1-\sigma}{\mathbb{B}(\sigma)}h(t) + \frac{\sigma}{\mathbb{B}(\sigma)\Gamma(\sigma)}\int_0^t (t-\tau)^{\sigma-1}h(\tau)d\tau. \tag{6} $$

Theorem 1 *(Banach's contraction principle (Cakan and Ozdemir, 2014).* Let **B** be a Banach space, and **K** a nonempty closed subset of **B**. If $\Pi : \mathbf{K} \longrightarrow \mathbf{K}$ is a contraction mapping, then there exists a unique fixed point of **B**.

Theorem 2 (*Generalized Gronwall's inequality (Jarad et al., 2018)) Let, $u(t)\left(1-\frac{1-\sigma}{\mathbb{B}(\sigma)}v(t)\right)^{-1}$, a nonnegative, non-decreasing and locally integrable function on $[a,b)$ and $\frac{\sigma v(t)}{\mathbb{B}(\sigma)}\left(1-\frac{1-\sigma}{\mathbb{B}(\sigma)}v(t)\right)^{-1}$, a non-negative and bounded in $[a,b)$. If $\phi(t)$ is a nonnegative and locally integrable function in $[a,b)$ with,*

$$ \phi(t) \le u(t) + v(t)({}^{AB}I_t^{\sigma}\phi)(t), \tag{7} $$

then

$$ \phi(t) \le \frac{u(t)\mathbb{B}(\sigma)}{\mathbb{B}(\sigma)-(1-\sigma)v(t)} E_{\sigma}\left(\frac{\sigma\, v(t)(t-a)^{\sigma}}{\mathbb{B}(\sigma)-(1-\sigma)v(t)}\right). \tag{8} $$

Theorem 3 (Krasnoselskii's fixed point theorem (Cakan and Ozdemir, 2014)). *Let* **Y** *be a closed convex subset of a Banach space* **B** *and let* Π_1, Π_2 *be operators on* **Y** *satisfying the following conditions:*

i. $\Pi_1 w^* + \Pi_2 w^* \in \mathbf{Y}$ for every $w^* \in \mathbf{Y}$;
ii. Π_1 is contraction, that is, there exists $\Xi > 0$ such that $\|\Pi_1 w^* - \Pi_1 w^{**}\| \leq \Xi \|w^* - w^{**}\|$ for every $w^*, w^{**} \in \mathbf{Y}$;
iii. Π_2 is a relatively compact subset of **B**.
Then there exists $w \in \mathbf{Y}$ such that $\Pi_1 w + \Pi_2 w = w$.

Helminth transmission model with integer order derivative

In the present section, we recall the integer order helminth transmission model formulated in Lambura et al. (2020). In their model, the total human population denoted by $N(t)$ is divided into four mutually exclusive epidemiological compartments based on dynamics of helminth infection. These compartments consist of susceptible $S(t)$, the exposed $E(t)$, infectious $I(t)$ and recovered $R(t)$ individuals, such that,

$$N(t) = S(t) + E(t) + I(t) + R(t).$$

The susceptible compartment consists of those individuals who are not infected by the helminth parasite but have the tendency of being infected by the disease under certain conditions. The exposed compartment consists of those individuals who have been infected by the disease but are not capable of transmitting the disease to others, possibly due to the level of concentration of the infection in them. The infectious compartment consists of individuals who have been infected by the helminth parasite and are capable of infecting others, owing to the presence of a higher concentration of the infection in them. The recovered compartment is made up of those individuals who have recovered from the infection. The model also incorporates an additional compartment denoted by $M(t)$ which consists of the concentration of helminth parasites within an environment at any given time. The model considered in Lambura et al. (2020) is given as,

$$\begin{cases} \frac{dS}{dt} = bN + \gamma R - (\mu + \lambda)S\ , \\ \frac{dE}{dt} = \lambda S - (\mu + \rho)E, \\ \frac{dI}{dt} = \rho E - (d + \mu + q + \varepsilon)I, \\ \frac{dR}{dt} = qI - (\mu + \gamma)R, \\ \frac{dM}{dt} = \varepsilon I - \mu_M M, \end{cases} \tag{9}$$

with initial data,

$$S(0) = S_0 \geq 0, E(0) = E_0 \geq 0, I(0) = I_0 \geq 0, R(0) = R_0 \geq 0, M(0) = M_0 \geq 0. \tag{10}$$

The parameters in the model (9) are positive constants and are described in Table 1.

BASIC ANALYSIS OF THE INTEGER ORDER MODEL

Here, we will report some important dynamical properties already presented in Lambura et al. (2020) concerning solutions of the integer order model (9) with respect to the initial conditions (10). To this end, we set,

Table 1. Description of model parameters.

Parameter	Description
b	Recruitment rate in to the susceptible class
ρ	progression rate from the exposed to infective class
ε	rate at which an infected individual contaminates the environment
q	progression rate from the infective to recovered class
γ	Immunity waning rate for recovered individuals
μ	natural mortality rate
d	Helminth induced death rate
μ_M	natural mortality rate for parasitic worm

$$s = \frac{S}{N}, \qquad e = \frac{E}{N}, \qquad i = \frac{I}{N}, \qquad r = \frac{R}{N}, \qquad m = \frac{M}{K},$$

to obtain the following associated normalize model:

$$\begin{cases} \frac{ds}{dt} = b + \gamma R - (b + \lambda^* - (d+\varepsilon)i)s\,, \\ \frac{de}{dt} = \lambda^* s - (b + \rho - (d+\varepsilon)i)e, \\ \frac{di}{dt} = \rho e - (b + d + q + \varepsilon - (d+\varepsilon)i)i, \\ \frac{dr}{dt} = qi - (b + \gamma - (d+\varepsilon)i)r, \\ \frac{dm}{dt} = \varepsilon i - \mu_M m, \end{cases} \qquad (11)$$

where term $\lambda^* = \frac{\beta m}{1+m}$ appearing in the first equation of (11) denotes the force of infection associated with the normalized model. Moreover, it is easy to see that,

$$s(t) + e(t) + i(t) + r(t) = 1. \qquad (12)$$

Then, in place of (9), it is convenient to study the dynamics of the normalized model (11) (Lambura et al., 2020) with respect to the initial conditions,

$$s(0) = s_0 \geq 0, e(0) = e_0 \geq 0, i(0) = i_0 \geq 0, r(0) = r_0 \geq 0, m(0) = m_0 \geq 0. \qquad (13)$$

By adding all equations in (11), the dynamics of the normalized model satisfies the following equality:

$$\frac{dN}{dt} = (b + \mu - (d+\varepsilon)i)N. \qquad (14)$$

To ensure epidemiologically well-posedness of the normalized transmission model (11), one must establish that all the system state variables are non-negative for all time. That is, solutions of the model (11) with non-negative initial data will remain non-negative for all time $t > 0$.

Lemma 2 *Let* $\Omega_1 = \{(s,e,i,r) \in \mathbb{R}^4 \mid 0 < s+e+i+r \leq 1\}$ *and* $\Omega_2 = \{m \in \mathbb{R} \mid 0 < m \leq 1\}$. *Then, the region* $\Omega = \Omega_1 \times \Omega_2 \in \mathbb{R}^4 \times \mathbb{R}$ *is positively invariant and attracting with respect to the model (12) and non-negative solutions exists for all time* $t > 0$.

Proof see Lambura et al. (2020).

Next, the equilibrium points of the normalized model (11) are steady state solutions obtained by setting the,
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$$\frac{ds}{dt} = \frac{de}{dt} = \frac{di}{dt} = \frac{dr}{dt} = \frac{dm}{dt} = 0.$$

Clearly, the normalized model (11) admits two equilibrium points, which are highlighted as follows:

Helminth free equilibrium points (HFE): The Helminth free equilibrium point refers to the equilibrium point of the model (12) in the absence of helminth disease within the population. In this case $m = 0$ and hence $e = 0,\ i = 0,\ r = 0$. Thus, by setting,

$$\frac{ds}{dt} = \frac{de}{dt} = \frac{di}{dt} = \frac{dr}{dt} = \frac{dm}{dt} = 0$$

we have the HFE as,

$$\mathbf{E_{HFE}} = \{1,0,0,0,0\}. \qquad (15)$$

Helminth endemic free equilibrium points (HEE): The Helminth endemic equilibrium point refers to the equilibrium point of the model (9) when helminth disease is endemic within the population. In this case $m \neq 0$ and hence $e \neq 0, i \neq 0, r \neq 0$. Thus, by setting,

$$\frac{ds}{dt} = \frac{de}{dt} = \frac{di}{dt} = \frac{dr}{dt} = \frac{dm}{dt} = 0$$

together with the above assumptions, we have the HEE as,

$$\mathbf{E_{HEE}} = \{s^{\#}(t), e^{\#}(t), i^{\#}(t), r^{\#}(t), m^{\#}(t)\}, \qquad (16)$$

Where,

$$\begin{cases} s^{\#}(t) = \dfrac{b(\gamma + z^*)(\rho + z^*)(d + q + z^* + \varepsilon)}{\Theta_3 + (z^*)^3(\lambda^* + \Theta_1) + (z^*)^2(\lambda^*\Theta_1 + \rho(\gamma + \Theta_2) + \gamma\Theta_2) + z^*\Theta_4 + (z^*)^4}, \\ e^{\#}(t) == \dfrac{b\lambda(\gamma + z^*)(\Theta_2 + z^*)}{(\gamma + z^*)(\lambda^* + z^*)(\rho + z^*)(\Theta_2 + z^*) - \gamma\lambda^* q\rho}, \\ i^{\#}(t) == \dfrac{b\lambda^*\rho(\gamma + z^*)}{\Theta_3 + (z^*)^3(\lambda^* + \Theta_1) + (z^*)^2(\lambda^*\Theta + \rho(\gamma + \Theta_2) + \gamma\Theta_2) + z^*\Theta_4 + (z^*)^4}, \\ r^{\#}(t) = \dfrac{b\lambda^* q\rho}{\Theta_3 + (z^*)^3(\lambda^* + \Theta_1) + (z^*)^2(\lambda^*\Theta_1 + \rho(\gamma + \Theta_2) + \gamma\Theta_2) + z^*\Theta_4 + (z^*)^4}, \\ m^{\#}(t) = \dfrac{b\lambda^*\rho\varepsilon(\gamma + z^*)}{\mu_M(\Theta_3 + (z^*)^3(\lambda^* + \Theta_1) + (z^*)^2(\lambda^*\Theta_1 + \rho(\gamma + \Theta_2) + \gamma\Theta_2) + z^*\Theta_4 + (z^*)^4)}. \end{cases} \qquad (17)$$

In (17), we used the following notations for the sake of computational convenience $z^* = b - (d + \varepsilon)i^{\#}$, $\Theta_1 = \gamma + d + q + \varepsilon$, $\Theta_2 = d + q + \varepsilon$, $\Theta_3 = \gamma\lambda^*\rho(d + \varepsilon)$, $\Theta_4 = \lambda^*(\rho(\gamma + \Theta_2) + \gamma\Delta_2) + \gamma\rho\Theta_2$. Furthermore, by the next generation approach due to Diekmann et al (2000), the basic reproduction number of the model is given by,

$$\mathcal{R}_0 = \frac{\rho\varepsilon\beta}{(b+\rho)(d+b+q+\varepsilon)\mu_M}.$$

Here, the fraction $\frac{\beta}{(d+b+q+\varepsilon)}$ denotes the average number of susceptible individuals being infected during the infectious period; $\frac{\rho}{(b+\rho)}$ is the proportion of individuals that survives the latent period and $\frac{\varepsilon}{\mu_M}$ represents the fraction of parasites diminished from the environment.

THE FRACTIONAL ORDER HELMINTH TRANSMISSION MODEL

The importance of integer order epidemic models in mathematical epidemiology cannot be ignored. However, these integer order epidemic models have certain number of limitations. Some of the limitations include the non-existence of memory or nonlocal effects and the inability to capture crossover behavior of physical or biological processes. To overcome these limitations, the fractional differential operators are incorporated in the modelling of biological systems because these operators take into account memory effects and the crossover behavior of the model. Therefore, to explore the helminth transmission model (9) more realistically, in the framework of fractional differential operators in the ABC sense, we replace the classical derivative by the fractional order in ABC derivative. Thus, the fractional helminth transmission model with the nonlocal kernel is given as:

$$\begin{cases} {}^{ABC}D_t^{\sigma}S(t) = bN + \gamma R - (\mu+\lambda)S\,, \\ {}^{ABC}D_t^{\sigma}E(t) \ = \lambda S - (\mu+\rho)E, \\ {}^{ABC}D_t^{\sigma}I(t) = \rho E - (d+\mu+q+\varepsilon), \\ {}^{ABC}D_t^{\sigma}R(t) \ = qI - (\mu+\gamma)R, \\ {}^{ABC}D_t^{\sigma}M(t) \ = \varepsilon I - \mu_M M \end{cases} \tag{18}$$

with association initial data:

$$S(0) = S_0 \geq 0, E(0) = E_0 \geq 0, I(0) = I_0 \geq 0, R(0) = R_0 \geq 0, M(0) = M_0 \geq 0. \tag{19}$$

Clearly, the fractional order model (10) generalizes the classical integer order model.

QUALITATIVE ANALAYASIS OF THYE FRACTIONAL MODEL

Existence and uniqueness of solutions

We establish the existence and uniqueness of solutions to the fractional model (18)-(19) using the approach of fixed point theory. The model stability is discussed in the sense of Ulam-Hyers. To this end, we set,

$$\aleph(t) := \begin{pmatrix} S(t) \\ E(t) \\ I(t) \\ R(t) \\ M(t) \end{pmatrix}, \quad \aleph(0) := \begin{pmatrix} S(0) \\ E(0) \\ I(0) \\ R(0) \\ M(0) \end{pmatrix} \quad \text{and} \quad \mathcal{F}\big(t,\aleph(t)\big) := \begin{pmatrix} \Psi_1\big(t,S(t),E(t),I(t),R(t),M(t)\big) \\ \Psi_2\big(t,S(t),E(t),I(t),R(t),M(t)\big) \\ \Psi_3\big(t,S(t),E(t),I(t),R(t),M(t)\big) \\ \Psi_4\big(t,S(t),E(t),I(t),R(t),M(t)\big) \\ \Psi_5\big(t,S(t),E(t),I(t),R(t),M(t)\big) \end{pmatrix}$$

Where,

$$\begin{cases}\Psi_1\big(t,S(t),E(t),I(t),R(t),M(t)\big)=bN+\gamma R-(\mu+\lambda)S,\\ \Psi_2\big(t,S(t),E(t),I(t),R(t),M(t)\big)=\lambda S-(\mu+\rho)E,\\ \Psi_3\big(t,S(t),E(t),I(t),R(t),M(t)\big)=\rho E-(d+\mu+q+\varepsilon)I,\\ \Psi_4\big(t,S(t),E(t),I(t),R(t),M(t)\big)=qI-(\mu+\gamma)R,\\ \Psi_5\big(t,S(t),E(t),I(t),R(t),M(t)\big)=\varepsilon I-\mu_m M.\end{cases} \tag{20}$$

represents the right hand side of each equation in the system (18). Then the fractional model (18)-(19) can be rewritten in the following compact form:

$$\begin{cases}{}^{ABC}D_t^{\sigma}\aleph(t)=\mathcal{F}\big(t,\aleph(t)\big), \quad t\in[0,T], \qquad 0<\sigma\leq 1,\\ \aleph(0)=\aleph_0\geq 0.\end{cases} \tag{21}$$

Based on Lemma 1, the fractional IVP (21) admits the following equivalent integral representation:

$$\aleph(t)=\aleph_0+\frac{1-\sigma}{\mathbb{B}(\sigma)}\mathcal{F}\big(t,\aleph(t)\big)+\frac{\sigma}{\mathbb{B}(\sigma)\Gamma(\sigma)}\int_0^t \mathcal{F}\big(\omega,\aleph(\omega)\big)\,(t-\omega)^{\sigma-1}d\omega. \tag{22}$$

Using (24), the operators $\mathbf{F},\mathbf{G}:\mathbf{Z}\longrightarrow\mathbf{Z}$ are defined as follows,

$$\begin{cases}\mathbf{F}[\aleph(t)]=\aleph_0+\dfrac{1-\sigma}{\mathbb{B}(\sigma)}\mathcal{F}\big(t,\aleph(t)\big),\\ \mathbf{G}[\aleph(t)]=\dfrac{\sigma}{\mathbb{B}(\sigma)\Gamma(\sigma)}\displaystyle\int_0^t \mathcal{F}\big(\omega,\aleph(\omega)\big)\,(t-\omega)^{\sigma-1}d\omega.\end{cases} \tag{23}$$

Further, let $T>0$ $(0\leq t\leq T<\infty)$ and $\mathbf{Z}=C([0,T],\mathbb{R}^5)$ a Banach space of continuous functions $\aleph:[0,T]\longrightarrow\mathbb{R}^5$ endowed with the norm $\|\aleph\|=\max_{t\in[0,T]}\{|\aleph(t)|:\aleph\in\mathbf{Z}\}$, where $\max_{t\in[0,T]}|\aleph(t)|=\max_{t\in[0,T]}(|S(t)|+|E(t)|+|I(t)|+|R(t)|+|M(t)|)$ and $S(t),E(t),I(t),R(t),M(t)\in C([0,T])$. In order to establish the existence and uniqueness of solutions to the fractional problem (18)-(19), we will assume that the following hypotheses are satisfied by the nonlinear function $\mathcal{F}:[0,T]\times\mathbb{R}^5\longrightarrow\mathbb{R}^5$ appearing in the corresponding fractional IVP (21):

(H1) (**Lipschitz condition**) With respect to the continuity of $\mathcal{F}$, there exists a constant $K_{\mathcal{F}}>0$ such that for each $\aleph_1,\aleph_2\in\mathbb{R}^5$,

$$\big|\mathcal{F}\big(t,\aleph_1(t)\big)-\mathcal{F}\big(t,\aleph_2(t)\big)\big|\leq K_{\mathcal{F}}|\aleph_1(t)-\aleph_2(t)|. \tag{24}$$

(H2) **(Linear growth bound)** There exists constants $\mu_{\mathcal{F}}>0$, $\eta_{\mathcal{F}}>0$ such that,

$$|\mathcal{F}(t,\aleph(t))|\leq\mu_{\mathcal{F}}|\aleph|+\eta_{\mathcal{F}}. \tag{25}$$

Theorem 4 *Under the hypothesis **(H1)-(H2),** the fractional IVP integral equation (21) has at least one solution if* $\frac{(1-\sigma)}{\mathbb{B}(\sigma)}K_{\mathcal{F}}<1$. *Consequently, the considered fractional model (18)-(19) possesses at least one solution.*

Proof: Next, for $t\in[0,T]$ we introduce the fixed point operator $\Pi:\mathbf{Z}\longrightarrow\mathbf{Z}$ defined by

$$\Pi[\aleph(t)]=\aleph_0+\frac{1-\sigma}{\mathbb{B}(\sigma)}\mathcal{F}\big(t,\aleph(t)\big)+\frac{\sigma}{\mathbb{B}(\sigma)\Gamma(\sigma)}\int_0^t \mathcal{F}\big(\omega,\aleph(\omega)\big)\,(t-\omega)^{\sigma-1}d\omega. \tag{26}$$

Clearly, $\Pi[\aleph(t)] = \mathbf{F}[\aleph(t)] + \mathbf{G}[\aleph(t)]$. Let $\mathbf{B}_\lambda = \{\aleph \in \mathbf{Z} : \|\aleph\| \leq \lambda, \ \lambda > 0\}$ be closed convex subset of $\mathbf{Z}$. Then we establish the proof of the theorem in the following three steps:

STEP 1: We need to show that $(\Pi\mathbf{B}_\lambda) \subset \mathbf{B}_\lambda$ for every $\aleph \in \mathbf{Z}$ and $t \in [0, T]$. By using hypothesis **(H2)**, we have,

$$\begin{aligned}
\|\Pi[\aleph(t)]\| &\leq \max_{t\in[0,T]} \left\{|\aleph_0| + \frac{1-\sigma}{\mathbb{B}(\sigma)}\left|\mathcal{F}\big(t,\aleph(t)\big)\right| + \frac{\sigma}{\mathbb{B}(\sigma)\Gamma(\sigma)}\int_0^t (t-\omega)^{\sigma-1}\left|\mathcal{F}\big(\omega,\aleph(\omega)\big)\right| d\omega \right\} \\
&= \left\{|\aleph_0| + \frac{1-\sigma}{\mathbb{B}(\sigma)}\left(\mu_{\mathcal{F}} \max_{t\in[0,T]}|\aleph| + \eta_{\mathcal{F}}\right) + \frac{\sigma}{\mathbb{B}(\sigma)\Gamma(\sigma)}\int_0^t (t-\omega)^{\sigma-1}\left(\mu_{\mathcal{F}} \max_{t\in[0,T]}|\aleph| + \eta_{\mathcal{F}}\right) d\omega \right\} \\
&\leq \left\{|\aleph_0| + \frac{1-\sigma}{\mathbb{B}(\sigma)}(\mu_{\mathcal{F}}\|\aleph\| + \eta_{\mathcal{F}}) + \frac{\sigma}{\mathbb{B}(\sigma)\Gamma(\sigma)}\int_0^t (t-\omega)^{\sigma-1}(\mu_{\mathcal{F}}\|\aleph\| + \eta_{\mathcal{F}}) d\omega \right\} \\
&= |\aleph_0| + \frac{1-\sigma}{\mathbb{B}(\sigma)}(\mu_{\mathcal{F}}\lambda + \eta_{\mathcal{F}}) + \frac{T^\sigma}{\mathbb{B}(\sigma)\Gamma(\sigma)}(\mu_{\mathcal{F}}\lambda + \eta_{\mathcal{F}}) \\
&= |\aleph_0| + \left(\frac{1-\sigma}{\mathbb{B}(\sigma)} + \frac{T^\sigma}{\mathbb{B}(\sigma)\Gamma(\sigma)}\right)\eta_{\mathcal{F}} + \left(\frac{1-\sigma}{\mathbb{B}(\sigma)} + \frac{T^\sigma}{\mathbb{B}(\sigma)\Gamma(\sigma)}\right)\mu_{\mathcal{F}}\lambda \\
&= \Theta_1 + \lambda\Theta_2 \leq \lambda,
\end{aligned}$$

where $\Theta_1 := |\aleph_0| + \left(\frac{1-\sigma}{\mathbb{B}(\sigma)} + \frac{T^\sigma}{\mathbb{B}(\sigma)\Gamma(\sigma)}\right)\eta_{\mathcal{F}} < 1$ and $\Theta_2 := \left(\frac{1-\sigma}{\mathbb{B}(\sigma)} + \frac{T^\sigma}{\mathbb{B}(\sigma)\Gamma(\sigma)}\right)\mu_{\mathcal{F}} < 1$. Thus, with $\lambda = \frac{\Theta_1}{1-\Theta_2}$, it holds that $(\Pi\mathbf{B}_\lambda) \subset \mathbf{B}_\lambda$.

STEP 2: In this step, we show that the operator **F** is a contraction mapping. To this end, let $\aleph, \widetilde{\aleph} \in \mathbf{B}_\lambda$. Then by the hypothesis **(H1)** we have,

$$\begin{aligned}
\left\|\mathbf{F}[\aleph(t)] + \mathrm{F}[\widetilde{\aleph}(t)]\right\| &= \max_{t\in[0,T]} \frac{1-\sigma}{\mathbb{B}(\sigma)}\left|\mathcal{F}\big(t,\aleph(t)\big) - \mathcal{F}\big(t,\widetilde{\aleph}(t)\big)\right| \\
&\leq \frac{1-\sigma}{\mathbb{B}(\sigma)} K_{\mathcal{F}} \max_{t\in[0,T]}\left|\aleph(t) - \widetilde{\aleph}(t)\right| \\
&\leq \frac{1-\sigma}{\mathbb{B}(\sigma)} K_{\mathcal{F}}\left\|\aleph(t) - \widetilde{\aleph}(t)\right\|.
\end{aligned}$$

Thus, under the condition $\frac{1-\sigma}{\mathbb{B}(\sigma)} K_{\mathcal{F}} < 1$, the operator **F** is a contraction mapping.

STEP 3: Next, we show that **G** is relatively compact for any $\aleph \in \mathbf{B}_\lambda$. To achieve this, we show that **G** is continuous, uniformly bounded, and equi-continuous. Clearly, the operator **G** is continuous since $\mathcal{F}\big(\cdot,\aleph(\cdot)\big)$; then is continuous. Now, for $\aleph \in \mathbf{B}_\lambda$ we have,

$$\begin{aligned}
\|\mathbf{G}[\aleph(t)]\| &\leq \max_{t\in[0,T]} \frac{\sigma}{\mathbb{B}(\sigma)\Gamma(\sigma)}\int_0^t (t-\omega)^{\sigma-1}\left|\mathcal{F}\big(\omega,\aleph(\omega)\big)\right| d\omega \\
&\leq \frac{\sigma}{\mathbb{B}(\sigma)\Gamma(\sigma)}\int_0^t (t-\omega)^{\sigma-1}\left(\mu_{\mathcal{F}} \max_{t\in[0,T]}|\aleph| + \eta_{\mathcal{F}}\right) d\omega \\
&\leq \frac{\sigma}{\mathbb{B}(\sigma)\Gamma(\sigma)}\int_0^t (t-\omega)^{\sigma-1}(\mu_{\mathcal{F}}\|\aleph\| + \eta_{\mathcal{F}}) d\omega \\
&\leq \frac{T^\sigma}{\mathbb{B}(\sigma)\Gamma(\sigma)}(\mu_{\mathcal{F}}\lambda + \eta_{\mathcal{F}}).
\end{aligned}$$

This implies that $\mathbf{G}$ is uniformly bounded on $\mathbf{B}_\lambda$. Finally, we show that $\mathbf{G}$ equicontinuous. Let $\aleph \in \mathbf{B}_\lambda$ and $t_1, t_2 \in [0, T]$ such that $t_1 < t_2$. Then,

$$\begin{aligned}\|\mathbf{G}[\aleph(t_2)] - \mathbf{G}[\aleph(t_1)]\| &\leq \frac{\sigma}{\mathbb{B}(\sigma)\Gamma(\sigma)}\int_{t_1}^{t_2}(t_2-\omega)^{\sigma-1}\left|\mathcal{F}\big(\omega,\aleph(\omega)\big)\right|d\omega \\ &\quad + \frac{\sigma}{\mathbb{B}(\sigma)\Gamma(\sigma)}\int_0^{t_1}(t_1-\omega)^{\sigma-1}-(t_2-\omega)^{\sigma-1}\left|\mathcal{F}\big(\omega,\aleph(\omega)\big)\right|d\omega \\ &\leq \frac{2(t_2-t_1)^\sigma + (t_1^\sigma - t_2^\sigma)}{\mathbb{B}(\sigma)\Gamma(\sigma)}(\mu_{\mathcal{F}}\lambda + \eta_{\mathcal{F}})\end{aligned}$$

As $t_1 \longrightarrow t_2$ the right-hand side of the last inequality above tends to zero. Hence, $\mathbf{G}$ is uniformly continuous and bounded. Consequently, by Azela-Ascoli theorem, $\mathbf{G}$ is relatively compact and so it is as well completely continuous. Thus, by Theorem 1, we deduce that the fractional IVP problem (21) has at least one solution. Consequently, the fractional model (18)-(19) admits at least one solution.

Theorem 5 *Suppose that* ***(H1)*** *holds, then the fractional IVP (21) admits a unique solution if,*

$$\left(\frac{1-\sigma}{\mathbb{B}(\sigma)} + \frac{T^\sigma}{\mathbb{B}(\sigma)\Gamma(\sigma)}\right)K_{\mathcal{F}} < 1. \qquad (27)$$

Consequently, the considered fractional model (18)-(19) admits a unique solution.
Proof. Let $\aleph, \aleph' \in \mathbf{B}_\lambda$ and $t \in [0, T]$. Then, from the operator equation in (26), we have,

$$\begin{aligned}&\|\Pi[\aleph(t)] - \Pi[\aleph'(t)]\| \leq \|\mathbf{F}[\aleph(t)] - \mathbf{F}[\aleph'(t)]\| + \|\mathbf{G}[\aleph(t)] - \mathbf{G}[\aleph'(t)]\| \\ &\leq \frac{1-\sigma}{\mathbb{B}(\sigma)}\max_{t\in[0,T]}\left|\mathcal{F}\big(t,\aleph(t)\big) - \mathcal{F}\big(t,\aleph'(t)\big)\right| \\ &\quad + \frac{\sigma}{\mathbb{B}(\sigma)\Gamma(\sigma)}\int_0^t (t-\omega)^{\sigma-1}\max_{t\in[0,T]}\left|\mathcal{F}\big(t,\aleph(\omega)\big) - \mathcal{F}\big(t,\aleph'(\omega)\big)\right|d\omega \\ &\leq \left(\frac{1-\sigma}{\mathbb{B}(\sigma)} + \frac{T^\sigma}{\mathbb{B}(\sigma)\Gamma(\sigma)}\right)K_{\mathcal{F}}\|\aleph(t) - \aleph'(t)\|.\end{aligned}$$

Clearly, if $\left(\frac{1-\sigma}{\mathbb{B}(\sigma)} + \frac{T^\sigma}{\mathbb{B}(\sigma)\Gamma(\sigma)}\right)K_{\mathcal{F}} < 1$, then the operator Π admits a unique fixed point, by Banach contraction principle. Therefore, the fractional IVP (21) admits a unique solution. Consequently, the fractional model (18)-(19) admits a unique solution.

Stability analysis (Ulam-Hyers stability)

Theorem 6 *Suppose that* ***(H1)*** *is satisfied, then the zero solution of the fractional IVP (21) is stable and bounded if,*

$$\left(\frac{1-\sigma}{\mathbb{B}(\sigma)} + \frac{T^\sigma}{\mathbb{B}(\sigma)\Gamma(\sigma)}\right)K_{\mathcal{F}} < 1.$$

Proof: In view of Lemma 1, we recall that the fractional IVP (21) admits a unique solution given by the integral equation (22). Set $\sup_{t\in[0,T]}\|\mathcal{F}(t,0)\| \coloneqq \lambda$. Then by **(H1)** we have,

$$\|\aleph(t)\| \leq \|\aleph_0\| + \frac{1-\sigma}{\mathbb{B}(\sigma)}\left\{\left\|\mathcal{F}\big(t,\aleph(t)\big) - \mathcal{F}(t,0)\right\| + \|\mathcal{F}(t,0)\|\right\}$$

$$+\frac{\sigma}{\mathbb{B}(\sigma)\Gamma(\sigma)}\int_0^t (t-\omega)^{\sigma-1}\left\{\left\|\mathcal{F}\big(\omega,\aleph(\omega)\big)-\mathcal{F}(\omega,0)\right\|+\|\mathcal{F}(\omega,0)\|\right\}d\omega$$
$$\leq \|\aleph_0\|+\frac{1-\sigma}{\mathbb{B}(\sigma)}(K_{\mathcal{F}}\|\aleph\|+\lambda)+\frac{T^\sigma}{\mathbb{B}(\sigma)\Gamma(\sigma)}\lambda+\frac{T^\sigma}{\mathbb{B}(\sigma)\Gamma(\sigma)}K_{\mathcal{F}}\|\aleph(t)\|$$
$$=\|\aleph_0\|+\left(\frac{1-\sigma}{\mathbb{B}(\sigma)}+\frac{T^\sigma}{\mathbb{B}(\sigma)\Gamma(\sigma)}\right)\lambda+\left(\frac{1-\sigma}{\mathbb{B}(\sigma)}+\frac{T^\sigma}{\mathbb{B}(\sigma)\Gamma(\sigma)}\right)K_{\mathcal{F}}\|\aleph(t)\|.$$

This implies,

$$\|\aleph(t)\|\leq\frac{\|\aleph_0\|+\left(\frac{1-\sigma}{\mathbb{B}(\sigma)}+\frac{T^\sigma}{\mathbb{B}(\sigma)\Gamma(\sigma)}\right)\lambda}{1-\left(\frac{1-\sigma}{\mathbb{B}(\sigma)}+\frac{T^\sigma}{\mathbb{B}(\sigma)\Gamma(\sigma)}\right)K_{\mathcal{F}}}:=\delta.$$

This implies $\|\aleph(t)\|\leq\delta,\ \delta>0$. Hence, the fractional IVP (21) is stable and bounded.

Definition 4 *The proposed fractional IVP (21) is said to be Ulam-Hyers stable if there exists a real number $C_{\mathcal{H}}=max(C_{\mathcal{H}1},C_{\mathcal{H}2},C_{\mathcal{H}3},C_{\mathcal{H}4},C_{\mathcal{H}5})^T>0$ such that the following statement is satisfied: For some $\varepsilon=max(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5)^T>0$ and each solution $\tilde{\aleph}\in\boldsymbol{Z}$ satisfying the inequality,*

$$\left\|{}^{ABC}D_t^\sigma\tilde{\aleph}(t)-\mathcal{H}\big(t,\tilde{\aleph}(t)\big)\right\|\leq\varepsilon, \qquad (28)$$

there exists a unique solution $\aleph\in\boldsymbol{Z}$ of (11) such that,

$$\left\|\aleph(t)-\tilde{\aleph}(t)\right\|<C_{\mathcal{F}}\varepsilon,\qquad t\in[0,T],$$

where

$$\tilde{\aleph}(t)=\begin{pmatrix}\tilde{S}(t)\\\tilde{E}(t)\\\tilde{I}(t)\\\tilde{R}(t)\\\tilde{M}(t)\end{pmatrix},\quad \tilde{\aleph}(0)=\begin{pmatrix}\tilde{S}(0)\\\tilde{E}(0)\\\tilde{I}(0)\\\tilde{R}(0)\\\tilde{M}(0)\end{pmatrix},\quad \mathcal{F}\big(t,\tilde{\aleph}(t)\big)=\begin{pmatrix}\Psi_1\big(t,\tilde{S}(t),\tilde{E}(t),\tilde{I}(t),\tilde{R}(t),\tilde{M}(t)\big)\\\Psi_2\big(t,\tilde{S}(t),\tilde{E}(t),\tilde{I}(t),\tilde{R}(t),\tilde{M}(t)\big)\\\Psi_3\big(t,\tilde{S}(t),\tilde{E}(t),\tilde{I}(t),\tilde{R}(t),\tilde{M}(t)\big)\\\Psi_4\big(t,\tilde{S}(t),\tilde{E}(t),\tilde{I}(t),\tilde{R}(t),\tilde{M}(t)\big)\\\Psi_5\big(t,\tilde{S}(t),\tilde{E}(t),\tilde{I}(t),\tilde{R}(t),\tilde{M}(t)\big)\end{pmatrix}$$

Furthermore, it is also said to be generalized Ulam-Hyers stable if there exists $\Theta_{\mathcal{H}}\in C(\mathbb{R}_+,\mathbb{R}_+)$ with $\Theta(0)=0$, any solution $\tilde{\aleph}(t)$ satisfying (28) and any unique solution $\aleph(t)$ of (21), the following inequality holds $\left\|\aleph(t)-\tilde{\aleph}(t)\right\|<\Theta_{\mathcal{H}}(\varepsilon)$.

Remark 1: Consider a small perturbation $g\in C([0,T],\mathbb{R})$ with $g(0)=0$ satisfying the following properties:

$|g(t)|\leq\varepsilon$, for $t\in[0,T]$.
${}^{ABC}D_t^\sigma\tilde{\aleph}(t)=\mathcal{F}\left(t,\tilde{\aleph}(t)\right)+g(t)$, for $t\in[0,T]$ where
$g(t)=\max(g_1(t),g_2(t),g_3(t),g_4(t),g_5(t))^T$.

Lemma 3 *The solution $\tilde{\aleph}\in\boldsymbol{Z}$ of the perturbed problem,*

$$\begin{cases} {}^{ABC}D_t^{\sigma}\tilde{\aleph}(t) = \mathcal{F}\left(t, \tilde{\aleph}(t)\right) + \Psi(t), \\ \tilde{\aleph}(0) = \tilde{\aleph}_0 \geq 0, \end{cases} \tag{29}$$

satisfies the inequality,

$$\left|\tilde{\aleph}(t) - \left(\tilde{\aleph}_0 + \frac{1-\sigma}{\mathbb{B}(\sigma)}\mathcal{F}\left(t, \tilde{\aleph}(t)\right) + \frac{\sigma}{\mathbb{B}(\sigma)\Gamma(\sigma)}\int_0^t (t-\omega)^{\sigma-1}\mathcal{F}\left(\omega, \tilde{\aleph}(\omega)\right) d\omega\right)\right| \leq \left(\frac{1-\sigma}{\mathbb{B}(\sigma)} + \frac{T^{\sigma}}{\Gamma(\sigma)\mathbb{B}(\sigma)}\right)\varepsilon.$$

Proof: With the help of Lemma 1, the solution of the perturbed problem (29) is given by,

$$\tilde{\aleph}(t) = \tilde{\aleph}_0 + \frac{1-\sigma}{\mathbb{B}(\sigma)}\left(\mathcal{F}\left(t, \tilde{\aleph}(t)\right) + g(t)\right) + \frac{\sigma}{\mathbb{B}(\sigma)\Gamma(\sigma)}\int_0^t \left(\mathcal{F}\left(t, \tilde{\aleph}(t)\right) + g(\omega)\right) (t-\omega)^{\sigma-1} d\omega.$$

Moreover, in view of the Remark 1 we have,

$$\begin{aligned} &\left|\tilde{\aleph}(t) - \left(\tilde{\aleph}_0 + \frac{1-\sigma}{\mathbb{B}(\sigma)}\mathcal{F}\left(t, \tilde{\aleph}(t)\right) + \frac{\sigma}{\mathbb{B}(\sigma)\Gamma(\sigma)}\int_0^t (t-\omega)^{\sigma-1}\mathcal{F}\left(\omega, \tilde{\aleph}(\omega)\right) d\omega\right)\right| \\ &= \left|\frac{1-\sigma}{\mathbb{B}(\sigma)} g(t) + \frac{\sigma}{\mathbb{B}(\sigma)\Gamma(\sigma)}\int_0^t (t-\omega)^{\sigma-1} g(\omega) d\omega\right| \\ &\leq \frac{1-\sigma}{\mathbb{B}(\sigma)}|g(t)| + \frac{\sigma}{\mathbb{B}(\sigma)\Gamma(\sigma)}\int_0^t (t-\omega)^{\sigma-1}|g(\omega)| d\omega \\ &\leq \left(\frac{1-\sigma}{\mathbb{B}(\sigma)} + \frac{T^{\sigma}}{\Gamma(\sigma)\mathbb{B}(\sigma)}\right)\varepsilon. \end{aligned}$$

Theorem 7 *Under assumptions of Theorem 6, the solution of the considered model (9)-(10) is Ulam-Hyers stable in **Z**.*

Proof: Let $\tilde{\aleph} \in \mathbf{Z}$ be the solution of (28) and $\aleph \in \mathbf{Z}$ is a solution of (21) with initial condition $\aleph(0) = \tilde{\aleph}(0) = \tilde{\aleph}_0$. Then, by an application of Lemma 1 on (21) together with the fact that $\aleph(0) = \tilde{\aleph}_0$, we have the integral equation,

$$\aleph(t) \leq \tilde{\aleph}_0 - \frac{1-\sigma}{\mathbb{B}(\sigma)}\mathcal{F}\big(t, \aleph(t)\big) - \frac{\sigma}{\mathbb{B}(\sigma)\Gamma(\sigma)}\int_0^t (t-\omega)^{\sigma-1}\mathcal{F}\big(\omega, \aleph(\omega)\big) d\omega. \tag{28}$$

Moreover, in view of (28), assumption **(H2)** and Lemma 3, we obtain,

$$\begin{aligned} \left|\tilde{\aleph}(t) - \aleph(t)\right| &\leq \left|\tilde{\aleph}(t) - \tilde{\aleph}_0 - \frac{1-\sigma}{\mathbb{B}(\sigma)}\mathcal{F}\big(t, \aleph(t)\big) - \frac{\sigma}{\mathbb{B}(\sigma)\Gamma(\sigma)}\int_0^t (t-\omega)^{\sigma-1}\mathcal{F}\big(\omega, \aleph(\omega)\big) d\omega\right| \\ &\leq \left|\tilde{\aleph}(t) - \tilde{\aleph}_0 - \frac{1-\sigma}{\mathbb{B}(\sigma)}\mathcal{F}\left(t, \tilde{\aleph}(t)\right) - \frac{\sigma}{\mathbb{B}(\sigma)\Gamma(\sigma)}\int_0^t (t-\omega)^{\sigma-1}\mathcal{F}\left(\omega, \tilde{\aleph}(\omega)\right) d\omega\right| \\ &+ \frac{1-\sigma}{\mathbb{B}(\sigma)}\left|\mathcal{F}\left(t, \tilde{\aleph}(t)\right) - \mathcal{F}\big(t, \aleph(t)\big)\right| \\ &+ \frac{\sigma}{\mathbb{B}(\sigma)\Gamma(\sigma)}\int_0^t \left|\mathcal{F}\left(\omega, \tilde{\aleph}(\omega)\right) - \mathcal{F}\big(\omega, \aleph(\omega)\big)\right| (t-\omega)^{\sigma-1} d\omega \\ &\leq \left(\frac{1-\sigma}{\mathbb{B}(\sigma)} + \frac{T^{\sigma}}{\Gamma(\sigma)\mathbb{B}(\sigma)}\right)\varepsilon + \frac{1-\sigma}{\mathbb{B}(\sigma)} K_{\mathcal{F}}\left|\tilde{\aleph}(t) - \aleph(t)\right| \end{aligned}$$

$$+\frac{\sigma}{\mathbb{B}(\sigma)\Gamma(\sigma)}K_{\mathcal{F}}\int_0^t\left|\tilde{\aleph}(t)-\aleph(t)\right|(t-\omega)^{\sigma-1}d\omega.$$
$$\leq\left(\frac{1-\sigma}{\mathbb{B}(\sigma)}+\frac{T^{\sigma}}{\Gamma(\sigma)\mathbb{B}(\sigma)}\right)\varepsilon+K_{\mathcal{F}}\left({}^{AB}I_t^{\sigma}\left|\tilde{\aleph}(t)-\aleph(t)\right|\right).$$

Applying Theorem 2 with $u(t)=\left(\frac{1-\sigma}{\mathbb{B}(\sigma)}+\frac{T^{\sigma}}{\Gamma(\sigma)\mathbb{B}(\sigma)}\right)\varepsilon$ and $v(t)=K_{\mathcal{F}}$, it is easy to see that the last inequality above implies,

$$\left|\tilde{\aleph}(t)-\aleph(t)\right|\leq\frac{\left(\frac{1-\sigma}{\mathbb{B}(\sigma)}+\frac{T^{\sigma}}{\Gamma(\sigma)\mathbb{B}(\sigma)}\right)\varepsilon\,\mathbb{B}(\sigma)}{\mathbb{B}(\sigma)-(1-\sigma)K_{\mathcal{F}}}E_{\sigma}\left(\frac{\sigma\,K_{\mathcal{F}}t^{\sigma}}{\mathbb{B}(\sigma)-(1-\sigma)K_{\mathcal{F}}}\right)\leq C_{\mathcal{F}}\varepsilon$$

Where,

$$C_{\mathcal{H}}=\frac{\left(\frac{1-\sigma}{\mathbb{B}(\sigma)}+\frac{T^{\sigma}}{\Gamma(\sigma)\mathbb{B}(\sigma)}\right)\mathbb{B}(\sigma)}{\mathbb{B}(\sigma)-(1-\sigma)K_{\mathcal{F}}}E_{\sigma}\left(\frac{\sigma\,K_{\mathcal{F}}T^{\alpha}}{\mathbb{B}(\sigma)-(1-\sigma)K_{\mathcal{F}}}\right).$$

This implies that the fractional IVP (21) is Ulam-Hyers stable. As a consequence, the fractional model (18)-(19) is Ulam-Hyers stable.

Corollary 1 Under the hypotheses of Theorem 7, if there exists $\Theta_{\mathcal{F}}:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ with $\Theta_{\mathcal{H}}(0)=0$, then the fractional IVP (21) is generalized Ulam–Hyers stabile. Consequently, the fractional model (18)-(19) is generalized Ulam–Hyers stabile.

Proof. Choosing $\Theta_{\mathcal{F}}(\varepsilon)=C_{\mathcal{F}}\varepsilon$ and $\Theta_{\mathcal{F}}(0)=0$, then from Theorem 7 we have,

$$\left|\tilde{\aleph}(t)-\aleph(t)\right|\leq\Theta_{\mathcal{F}}(\varepsilon).$$

CONCLUSION

In this paper, a time fractional model describing the transmission dynamics of helminth disease is considered. Firstly, a classical integer order model in the form of a system of nonlinear ordinary differential equations is formulated using a compartmentalized approach. Some basic properties such as non-negativity of solutions, invariant region as well as equilibrium points are investigated for the integer order model. Furthermore, we extend the integer order model to its generalized fractional order counterpart by incorporating the fractional derivative in the Atangana-Baleanu-Caputo sense. By employing a fixed point approach, the fractional model is shown to admit a unique solution. Additionally, using the generalized Gronwall inequality, the Ulam–Hyers and generalized Ulam–Hyers stability are established for the fractional model.

REFERENCES

Lambura, A. G., Mwanga, G. G., Luboobi, L., and Kuznetsov. D. (2020). Mathematical Model for Optimal Control of Soil-Transmitted Helminth Infection. Computational and Mathematical Methods in Medicine, 2020. Article ID 6721919, https://doi.org/10.1155/2020/6721919

Atangana, A., and Baleanu, D. (2016). New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model. Therm. Sci., 20: 763–785.

Caputo, M., and Mauro, F. (2015). A new definition of fractional derivative without singular kernel. Prog. Fract. Diff. Appl., 1: 1-13.

Jarad, F., Abdeljawad, T., and Hammouch, F. (2018). On a class of ordinary differential equations in the frame of Atagana-Baleanu derivative. Chaos Solitons Fractals, 117: 16-20.

Jennifer, K., and Jurg, U. (2008). Efficacy of Current Drugs Against Soil-Transmitted Helminth Infections: Systematic Review and Meta-analysis. Journal of American Medical Association, 299(16): 1937- 1938.

Pullan, R. L., Smith, J. L., Jasrasaria, R., and Brooker, S. J., (2014). Global numbers of infection and disease burden of soil transmitted helminth infections in 2010. Parasites and vectors 7(1): 37.

Kilbas, A. A., Srivastava, H., and Trujillo, J. (2006). Theory and Application of Fractional Differential Equations in: North Holland Mathematics Studies. Amsterdam: Elseveir.

Podlubny, I. (1999). Fractional Differential Equations, Mathematics in Science and Engineering, Academic Press, New York, NY, USA.

Anderson, R. M., and May, R. M. (1978). Regulation and stability of host-parasite population interactions: I. Regulatory processes. The Journal of Animal Ecology, 47: 219–247.

Anderson, R. M., and May, R. M. (1982a). Population dynamics of human helminth infections: control by chemotherapy. Nature, 297: 557–563.

Anderson, R. M., and May R. M. (1982b). Directly transmitted infectious diseases: control by vaccination, Science, 215: 1053–1060.

World Health Organization (2012). Accelerating work to overcome the global impact of neglected tropical diseases – a roadmap for implementation (No. WHO/HTM/ NTD/2012.1). Geneva. 7-22.

Michael, E., Malecela-Lazaro, M. N., Kabali, C., Snow, L. C., and Kazura, J. W. (2006). Mathematical models and lymphatic filariasis control: endpoints and optimal interventions, Trends in Parasitology. 22: 226–233.

Diekmann, O., and Heesterbeek, J. A. P. (2000). Mathematical Epidemiology of Infectious Diseases: Model Building, Analysis and Interpretation (Vol. 5). West Sussex: John Wiley & Sons.

Cakan, U., and Ozdemir, I. (2014). An application of Krasnoselskii fixed point theorem to some nonlinear functional integral equations. Nevsehir Bilim ve Teknoloji Dergisi, 3(2), 66-73.

Codella, J., Safdar, N., Heffernan, R., and Alagoz, O. (2015). An Agent-based Simulation Model for Clostridium difficile Infection Control. Med Decis Making; PubMed 35: 211–29.

Truscott, J.E., Turner, H.C., Farrell, S.H., Anderson, R.M. (2016). Soil-transmitted helminths: mathematical models of transmission, the impact of mass drug administration and transmission elimination criteria. Advancement in Parasitology. 94: 133-198.