Approximate solutions of stochastic bonhoeffer-van der pol oscillatory system

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This paper considers the numerical simulation of stochastic Bonhoeffer Van Der Pol oscillation problem (SBVD), using the variational iteration method (VIM) and homotopy perturbation method (HPM) to implement the VIM and HPM on the stochastic Bonhoeffer Van der Pol oscillator. The study shows that SBVD oscillator oscillates within the interval $[-4,4]$. This implies that there is self-sustaining oscillation in which the energy is fed into small oscillations and removed from large oscillations. However, the HPM generates oscillations that possess much more energy than the VIM. When $\varphi = 2.5$, the equation reduces to the simple harmonic oscillator. All computational framework of this research were performed using MAPLE 18 software.

Key words: Stochastic differential equation, variational iteration method, homotopy perturbation method, Van Der Pol oscillation, Bonhoeffer Van Der Pol oscillation

INTRODUCTION

Many wide range physical applications in science and technology are mostly nonlinear systems and are of great significance in the world of mathematical modeling. The Van-der Pol equation was first introduced by a Dutch electrical engineer, Balthazar Van der Pol (1889 - 1959), who initiated the urge for experimental dynamics in the laboratory. He gave the experimental equation to describe the characteristics of triode oscillations in electrical circuits in 1927 (Marios, 2006). However, the underlying mathematical model for the dynamical system is a well-known second order ordinary differential equation with third order nonlinearity

$$\ddot{U} + \beta(U^2 - 1)\dot{U} + U = 0 \qquad (1)$$

where β is the controllable oscillatory coefficient, and $\ddot{U}$, $\dot{U}$ and U are differential operators of various orders. Equation 1 is known as the (unforced) Van der Pol equation (Van der Pol, 1920). Equation 1 is a mere simple mathematical model for excess characteristics observed in the laboratory experiment. If $\beta \geq 1$, then the Van der Pol Equation 1 produces a relaxation oscillations (Van der Pol, 1920). This discovery has become the bedrock of modern theory of geometric singular perturbation, which plays a major role in the analysis of autonomous and non-autonomous Van der Pol oscillators. Van der Pol continued in this line of research and proposes yet another version of equation (1.1) with a periodic forcing term given as

$$\ddot{U} + \beta(U^2 - 1)\dot{U} + U = b\cos(\omega t) \qquad (2)$$

where $\ddot{U}$, $\dot{U}$, U, and β are as defined in Equation 1, ω represent the angular velocity of the oscillation in time t.

However, it was noted that systems modeled by Van der Pol oscillators are chaotic and possess a periodic behavior into high level of sensitivity, which rely solely on initial conditions (Cartlight and Littlewood, 1945). Basically, Van der Pol oscillators are critical in the enhancement, development and implementation of nonlinear dynamic systems.

The Bonhoeffer-Van der Pol oscillator is a piecewise version of the Balthazar-Van der Pol oscillator, which confirms the underlying principles of Van der Pol oscillator to a more explicit Hodgkin-Huxley model (Partitz and Lauterbom, 1980). Ebeling (1976) reported that the Bonhoeffer Van der Pol oscillator (BVDP) has been classified as a prototype model for excitable systems. According to Weinhein (1988), the BVDP oscillators are governed by large and small alternating amplitude excursion in the real time series. Thus, a BVDP oscillator is an explicit non-autonomous system given as (Benitez and Bolos, 2007):

$$\begin{cases} X' = \left[X - \frac{x^3}{3} - U + A\cos(2\pi t)\right] \\ Y' = d[X + c - bU] \end{cases} \quad (3)$$

where the real constants b, c and d are of great physical and biological importance. The term $A\cos(2\pi t)$ is a periodic forcement term.

An oscillator is mathematically described by a set of (nonlinear) differential equations and considered to be a (nonlinear) dynamical system (Guckenheiner and Holmes, 1983). The influence of noise on nonlinear dynamical systems has been an object of intense investigations (Moss and McClintock, 1987), since real systems like biological systems always have noise.

The phenomenon of transitions induced by external noise has led to a revival of the interest for the role of fluctuations in physical systems (Horsthemke and Lefever, 1984). For instance, in a multi-stable system which possesses several competing states of local stability, noise can be responsible for transition between these states. Recently "noisy" systems have also received considerable attention within the context of stochastic resonance. Grasman and Roerdink (1989) analyzed the Van der Pol relaxation oscillator with additive noise and reduced the problem of examining the period of the noisy oscillator to the analysis of the time necessary for a one-dimensional stochastic process to reach a boundary for the first time.

The behavior of the system in Equation 3 is chaotic and periodic with a high level of sensitivity depending on initial conditions. As such, numerical techniques have become more reliable source of solution to the BVDP oscillation (1.3). Popular numerical methods for the system in Equation 3 include the invariant method (Benitez and Bolos, 2009) and the Melnikov scheme (Rajaseker and Parthasarathy, 1992). However, these methods only seek the intersections between stable and unstable manifolds, and not the explicit behavior of the system in terms of it randomness. Thus, the system is stochastic, and can therefore be modeled as a stochastic differential equation.

A transformation of the system in Equation 3 into stochastic differential equation yields:

$$\begin{cases} dX_t^1 = \left[X_t^1 - \frac{1}{3}(X_t^1)^3 - X_t^2 + A\cos 2\pi t\right] + \varphi(t)dW_t \\ dX_t^2 = d[X_t^1 + c - bX_t^2]dt \end{cases} \quad (4)$$

where φ is called the noise intensity, b, c and d are real constants, (t) is the state of the system and W_t is called the white noise. The subscript, t, denotes time dependent. Thus, Equation 4 is called the stochastic Bonhoeffer-Van Der Pol Oscillator. A special kind of Equation 4 is the model firing of single neuron given as:

$$\begin{cases} dX_t^1 = \left[X_t^1 - \frac{1}{3}(X_t^1)^3 - X_t^2 + Z\right] + \varphi dW_t \\ dX_t^2 = -\frac{1}{d}[X_t^1 - bX_t^2 - c]dt \end{cases} \quad (5)$$

In this paper, the stochastic Bonhoeffer Van-der Pol oscillator will be solved numerically using the variational iteration method (VIM) and homotopy perturbation method (HPM). The VIM as an iterative scheme was first proposed by He (1998) for the numerical treatment of nonlinear analytic systems. Since then, the method has become an efficient solver of many Mathematical problems in various fields of science and technology such as biophysics, laser physics, population dynamics, engineering, marketing, and plasma physics. The method can also be used for mathematical models involving integro-differential equations (ordinary or fractional), differential equations (ordinary, partial, delay, algebraic and stochastic) initial and boundary value problems etc. (Mamadu and Njoseh, 2016; Mamadu and Njoseh, 2017; Njoseh and Mamadu, 2016). The choice of VIM for solving in Equation 4 lies in its simplicity in estimating the initial approximation. It is also programmable.

In like manner, the HPM was proposed by He (1997) for solving a wide range of physical problems. It is a merger of the standard homotopy and perturbation method. It has major applications in limit cycle analysis, bifurcation of linear and nonlinear problems, and nonlinear oscillations (He, 2008). The choice of the HPM for Equation 4 lies in the easy observation of convergence of the scheme through the comparison of the embedded parameter p of the various orders (Othman et al., 2010).

VARIATIONAL ITERATION METHOD (VIM)

VIM developed by He (1998), is a very reliable and effective technique for obtaining the analytical and numerical solution of linear and non-linear, homogenous and non-homogenous equations. The technique presents its results in rapidly convergent series which converge to a close form of the exact solution. A major advantage of the variational iteration method is that it gives solution without the use of the Adomian polynomials which are mostly used for non-linear cases. The solution obtained can also be assumed as its exact solution if the analytical solution is not given. Hence the error is given by:

$$E = |y_n(t) - y_{n+1}(t)| \qquad (6)$$

Likewise, when the exact is known, its form is given by:

$$E = |y(t) - y_n(t)|$$

where $y(t)$ is the exact solution and $y_n(t)$ is the approximate solution.

To discuss the basic idea of variational iteration method (VIM), consider the differential equation:

$$Ly(t) + Ny(t) = f(t), \qquad (7)$$

where L and N are linear and nonlinear operators respectively, and $f(t)$, the forcing term or non-homogenous term (He, 1999; 2007). The variational iteration method admits the use of correction functional in the form:

$$y_{n+1}(t) = y_n(t) + \int_0^t \lambda(s)(Ly_n(s) + N\tilde{y}_n(s) - f(s))\,ds \qquad (8)$$

where λ is the general lagrange multiplier and can be obtained via variational theory (He, 1999). Hence for complete evaluation of the variational iteration method (VIM) we need to determine the Lagrange multiplier and then substitute back into the correction functional. Also, $\tilde{y}(t)$ is called a restricted variable.

The general Lagrange multiplier can also be evaluated using the formula (Mamadu and Njoseh, 2016):

$$\lambda(s) = (-1)^n \frac{(s-t)^{n-1}}{(n-1)!},$$

where n is the highest occurring derivative.

The correction application of the variational iteration method in Equation 8 gives several approximations which converge to the exact solution. We now state the following theorem on convergence of VIM.

Theorem 2.1

For Banach spaces Y oppose the non-linear mapping $B:Y \to y$ satisfy

$$\|B[u] - B[\bar{u}]\| \le \alpha\|u - \bar{u}\|, u, \bar{u} \in y$$

For some constant: $\alpha < 1$. Then a has a unique fixed point. Furthermore, the sequence

$$u_{n+1} = \beta[y_n]$$

with arbitrary choice of $u_0 \in Y$, converges to the fixed point B_0,

$$\|u_i - \bar{u}_i\| \le \|u_1 - u_0\| \sum_{e-i=1}^{i-2} 2^e$$

Hence, for non-linear mapping

$$B[u] = u(y,t) + \int_0^t [l + u(y,t) + Nu(y,t) - f(y,l)\,dl]$$

Proof. For each $u, \bar{u} \in Y$, the $\lim_{n\to\infty}\|B[u] - B[\bar{u}]\|$ exist, where B is a non-linear mapping satisfying $B:Y \to y$. Now for each $p \in Y$, where $p = (u, \bar{u})$, we have

$$\|B[u]-B[\bar{u}]\|^2 =< B[u]-p, j(B[u]-p) > \quad (9)$$

$$= \alpha_n < u_n - p, j(u_n - p) > +(1-\alpha_n) < \tau_n u_n - p, j(u_n - p) > \quad (10)$$

$$\leq \alpha_n \|u_{n-1} - p\| \|u_n - p\| + (1-\alpha_n)\|B[u]-B[\bar{u}]\|^2, n > 0. \quad (11)$$

Simplifying, we have that,
$\|u_n - p\| \leq \|u_{n-1} - p\| \Rightarrow u_{n+1} = \beta[y_n]$.

Thus, the limit $\lim_{n\to\infty}\|B[u]-B[\bar{u}]\|$ exists, and so the sequence $\{u_n\}$ is bounded.

Next, it shows that for some fixed constant the sequence $\{u_n\}$ converges to a definite fixed point

$$\|u_i - \bar{u}_i\| \leq \|u_1 - u_0\| \sum_{e-i=1}^{i-2} 2^e .$$

In view of in Equation 10 and 11, we have

$\|u_n - \tau_n u_n\| = \alpha_n \|u_n - \tau_n u_n\| \to 0$ (as n tends to infinity). (12)

The study shows that $\{u_n\}$ converges to some point in y. In fact, it follows from Equation 12 that there exists a subsequence $\{u_r\} \subset \{u_n\}$ such that $\|u_r - \tau_r u_r\| \to 0$ as $n_r \to \infty, \tau_r u_r \to p$ and $u_r \to p$ (some point u_0).

Consequently,

$$\|p - T_p\| \leq \|p - u_r\| + \|u_n - pu_n\| + \|\tau_n u_n - T_n \tau_n\|$$

$$\leq (1+L)\|p - u_r\| + \|u_n - pu_n\| + \|\tau_n u_n - T_n \tau_n\|$$

$$= \|u_n - pu_n\| \leq \|u_{n+1} - u_n\| \sum_{j-i=1}^{i-2} 2^j \to 0.$$

This implies that $T_p = p$, since $u_n \to p$ and the $\lim_{n\to\infty}\|B[u]-B[\bar{u}]\|$ exists, we have that $u_n \to p$.

HOMOTOPY PERTURBATION METHOD

To describe the HPM, the study considers the generalized differential equation of the form (Njoseh and Mamadu, 2017):

$$Lu(t) + Nu(t) = f(t),$$

We now define the operator

$$L[u(t)] = f(t) - Nu(t),$$

where $L[u(t)] = u(t)$,

Next apply the homotopy of the form $H(u,p), p\epsilon[0,1]$ by $H(u,0) = f(u), \; H(u,1) = L(u)$ (13)

where $f(u)$, are the functional operators. We construct a convex homotopy of the form

$$H(u,p) = (1-p)f(u) + pL(u) = 0 \quad (14)$$

This homotopy satisfies (3.2) for $p = 0$ and $p = 1$. The embedding parameter p monotonically increases from 0 to 1 as the trivial solution $f(u) = 0$ continuity defined (He, 1999), to the original problem $L(u) = 0$. The HPM admits the use of the expansion

$$u = \sum_{n=0}^{\infty} P^n U_n , \quad (15)$$

and consequently

$$S = \lim_{P\to 1} \sum_{n=0}^{\infty} P^n U_n, \quad (16)$$

The series converge to the exact solution if such a solution exists. Substituting Equation 15 into 14, and equating the terms with like powers of the embedding parameter P we obtain the recurrent relations

$$P^0 : u_0(t) = f(t),$$

$$\vdots$$

$$P^{n+1} : u_{n+1} = Lu_n(t) + Nu_n(t), \quad n \geq 0$$

In the case of VIM the correction functional for the stochastic Bonhoeffer Van-der Pol oscillator, was corrected first, and then obtain the general Lagrange multiplier optimally via the variational theory. The solution follows by iterating on the derived recurrence relation for $n \geq 0$. For HPM, we only need to construct a convex homotopy for the stochastic Bonhoeffer Van-der Pol oscillator and solved the solution using the given initial conditions.

Theorem 3.1
Let L(X) denote

$$\begin{cases} dX_t^1 = \left[X_t^1 - \frac{1}{3}(X_t^1)^3 - X_t^2 + A\cos 2\pi t\right] + \varphi(t)dW_t \\ dX_t^2 = d[X_t^1 + c - bX_t^2]dt \end{cases}$$

in equation 3 such that

$$X_{n+1}^1(t) = X_{t,n}^1(t) + \int_0^t \lambda(s)\left((X_{s,n}^1 - X_0^1 - \int_0^t\left(X_s^1 - \frac{1}{3}(X_s^1)^3 - X_s^2 + A\cos 2\pi t\right)ds - \int_0^t \varphi(t)dW_t\right)ds,\ n \geq 0$$

$$X_{t,n+1}^2(t) = X_{t,n}^2(t) + \int_0^t \lambda(s)\left(d(X_{s,n}^2 - X_0^2 - \int_0^t(X_s^1 + c - bX_s^2)ds\right)ds$$

where b, c and d are real constants, (t) is called the state of the system and W_t is called the white noise.

Then, VIM converges if the following conditions are satisfied:

1. $(L(X) - L(Y), X - Y) \geq \alpha\|X - Y\|^2, \alpha > 0, u, v \in H$

2. For $\beta > 0,$ there exist $\alpha(\beta)$ such that $\|X\| \leq \beta, \|Y\| \leq \beta,$

Then,

$(L(X) - L(Y), u - v) \geq \alpha(\beta)\|X - Y\|\|\Omega\|, \Omega \in H.$

Proof:

Let $\alpha > 0, X, Y \in H$ such that

$(L(X) - L(Y), X - Y) = \int_0^t \lambda(s)([L(X) - L(Y)]\,ds, X - Y)$

Applying the Schwartz inequality, we get

$\left(\int_0^t \lambda(s)([L(X) - L(Y)]\,ds, X - Y)\right) \leq \alpha_1\|L(X) - L(Y)\|\|X - Y\|$

By the conventional use of the mean value theorem, we obtained

$\left(\int_0^t \lambda(s)([L(X) - L(Y)]\,ds, X - Y)\right) \geq K_2\|X - Y\|^2,$

$K_2 = 0.5\alpha_2\beta^2$

Hence,

$(L(X) - L(Y), X - Y) \geq \alpha\|X - Y\|^2$,

holds with $\alpha = 0.5\alpha_2\beta^2$.

Similarly, for $\beta > 0, \exists\, \alpha(\beta) > 0$ such that $\|X\| \leq \beta, \|Y\| \leq \beta, X, Y \in H,$

Then,

$(L(u) - L(v), q) = \left(\int_0^t \lambda(s)([L(X) - L(Y)]\,ds, X - Y)\right) \leq \beta^2\|X - Y\|\|q\|$

This completes the proof.

NUMERICAL EXAMPLES

The VIM for stochastic Bonhoeffer Van der Pol oscillator

To start off the VIM process, we rewrite the stochastic Bonhoeffer Van der Pol oscillator in Equation 4 as

$$X_t^1 = X_0^1 + \int_0^t\left(X_s^1 - \frac{1}{3}(X_s^1)^3 - X_s^2 + A\cos 2\pi t\right)ds + \int_0^t \varphi(t)dW_t \quad (17)$$

$$X_t^2 = X_0^2 + \int_0^t d(X_s^1 + c - bX_s^2)ds \quad (18)$$

Equations 17 and 18 are known as the stochastic integral Bonhoeffer Van der Pol oscillator.

Based on the description of the VIM, a correction functional for Equations 17 and 18 are given as

$$X_{t,n+1}^1(t) = X_{t,n}^1(t) + \int_0^t \lambda(s)\left((X_{s,n}^1 - X_0^1 - \int_0^t\left(X_s^1 - \frac{1}{3}(X_s^1)^3 - X_s^2 + A\cos 2\pi t\right)ds - \int_0^t \varphi(t)dW_t\right)ds,\ n \geq 0 \quad (19)$$

$$X_{t,n+1}^2(t) = X_{t,n}^2(t) + \int_0^t \lambda(s)\left(d(X_{s,n}^2 - X_0^2 - \int_0^t(X_s^1 + c - bX_s^2)ds\right)ds \quad (20)$$

The general Lagrange multiplier is given as:

$\lambda(s) = (-1)^n \frac{(s-t)^{n-1}}{(n-1)!} \Rightarrow \lambda(s) = -1$ (since $n = 1$).

Thus, using VIM, Equations 17 and 18 becomes

$$X_{t,n+1}^1(t) = X_{t,n}^1(t) - \int_0^t\left((X_{s,n}^1 - X_0^1 - \int_0^t\left(X_s^1 - \frac{1}{3}(X_s^1)^3 - X_s^2 + A\cos 2\pi t\right)ds - \int_0^t \varphi(t)dW_s\right)ds, \quad (21)$$

$$X_{t,n+1}^2(t) = X_{t,n}^2(t) - \int_0^t\left(d(X_{s,n}^2 - X_0^2 - \int_0^t(X_s^1 + c - bX_s^2)ds\right)ds \quad (22)$$

The HPM for stochastic Bonhoeffer Van der Pol oscillator

The study presents the homotopy perturbation method for handling the stochastic Bonhoeffer

Van der pol oscillator. First, Equation 4 was rewritten

$$X_t^1 = X_0^1 + \int_0^t \left(X_s^1 - \frac{1}{3}(X_s^1)^3 - X_s^2 + A\cos 2\pi t\right) ds + \int_0^t \varphi(t$$

$$X_t^2 = X_0^2 + \int_0^t d(X_s^1 + c - bX_s^2)ds$$

We now define the operator as:

$$L[u(x)] = X_0^1 + \int_0^t \left(X_s^1 - \frac{1}{3}(X_s^1)^3 - X_s^2 + A\cos 2\pi t\right) ds + \int_0^t \varphi(t)dW_t$$

$$L[v(x)] = X_0^2 + \int_0^t d(X_s^1 + c - bX_s^2)ds$$

where $L[u(t)] = X_t^1$, $L[v(t)] = X_t^2$.

Next, the homotopy of the form $H(u,p), H(v,p)\ p\epsilon[0,1,]$ was applied by:

$$H(u,0) = f(u),\ H(u,1) = L(u) \quad (23)$$

$$H(v,0) = f(v),\ H(v,1) = L(v) \quad (24)$$

where $f(u), f(v)$ are the functional operators. We construct a convex homotopy of the form

$$H(u,p) = (1-p)f(u) + pL(u) = 0 \quad (25)$$

and $H(v,p) = (1-p)f(v) + pL(v) = 0$ (26)

This homotopy satisfies (4.9) and (4.10) for $p = 0$ and $p = 1$. The embedding parameter p monotonically increases from 0 to 1 as the trivial solution $f(u) = f(v) = 0$ continuity defined (He, 1999), to the original problem $L(u) = 0$. The HPM admits the use of the expansion

$$u = \sum_{n=0}^{\infty} P^n U_n, \quad v = \sum_{n=0}^{\infty} P^n V_n \quad (27)$$

and consequently

$$S = \lim_{P\to 1} \sum_{n=0}^{\infty} P^n U_n, \qquad T = \lim_{P\to 1} \sum_{n=0}^{\infty} P^n V_n \quad (28)$$

The series converge to the exact solution if such a solution exists. Substituting Equations 27 and 28 into Equations 25 and 26, and equating the terms with like powers of the embedding parameter p we obtain the recurrent relation

$$P^0 : u_0(x) = X_0^1,$$

$$\vdots$$

$$P^{n+1} : u_{n+1} = \int_0^t \left(X_n^1 - \frac{1}{3}(X_n^1)^3 - X_n^2 + A\cos 2\pi t\right) ds + \int_0^t \varphi(t)dW_t,\ n \geq 0$$

$$P^0 : v_0(x) = X_0^2,$$

$$\vdots$$

$$P^{n+1} : v_{n+1} = \int_0^t d(X_n^1 + c - bX_n^2)ds,\ n \geq 0$$

Remark

The white noise is discretized using a MAPLE 18 aided random walk analysis encoded within the program.

Now, for $t \in [0,50]$ and $n \geq 0$, executing the VIM scheme Equations 21 and 22 with MAPLE 18 software with the parameters $\varphi = 2.5,\ b = 0,\ c = 0.5,\ d = 0.5$, the behaviour of the SBVD oscillator is shown in Figure 1. Similarly, with HPM with same parameters we obtained the behaviour of the SBVD oscillations in Figure 2.

RESULTS AND DISCUSSION

Implementing the VIM and HPM on the stochastic Bonhoeffer Van der pol oscillator, it shows that SBVD oscillator oscillates within the interval $-4 \leq X_{t,n+1}^1(t), X_{t,n+1}^2(t) \leq 4$. This implies that there is self-sustaining oscillations in which the energy is fed into small oscillations and removed from large oscillations. However, the HPM generates oscillations that possess much more energy in self-sustenance, in which the energy is fed into small oscillations and removed from large oscillations. When $\varphi = 2.5$, the equation reduces to the simple harmonic oscillator.

Conclusion

It is important to note that for every mathematical formulations or constructions, many analytic methods are difficult to resolve in real sense. Thus, our results have shown that the iterative schemes VIM and HPM are efficient solvers of the explicit non-autonomous system of the Bonhoeffer Van-der Pol oscillator. On the basis of our analysis and computation, we conclude that the VIM and HPM are good iterative schemes for the solution of the explicit non-autonomous system of the Bonhoeffer Van-der Pol oscillator.

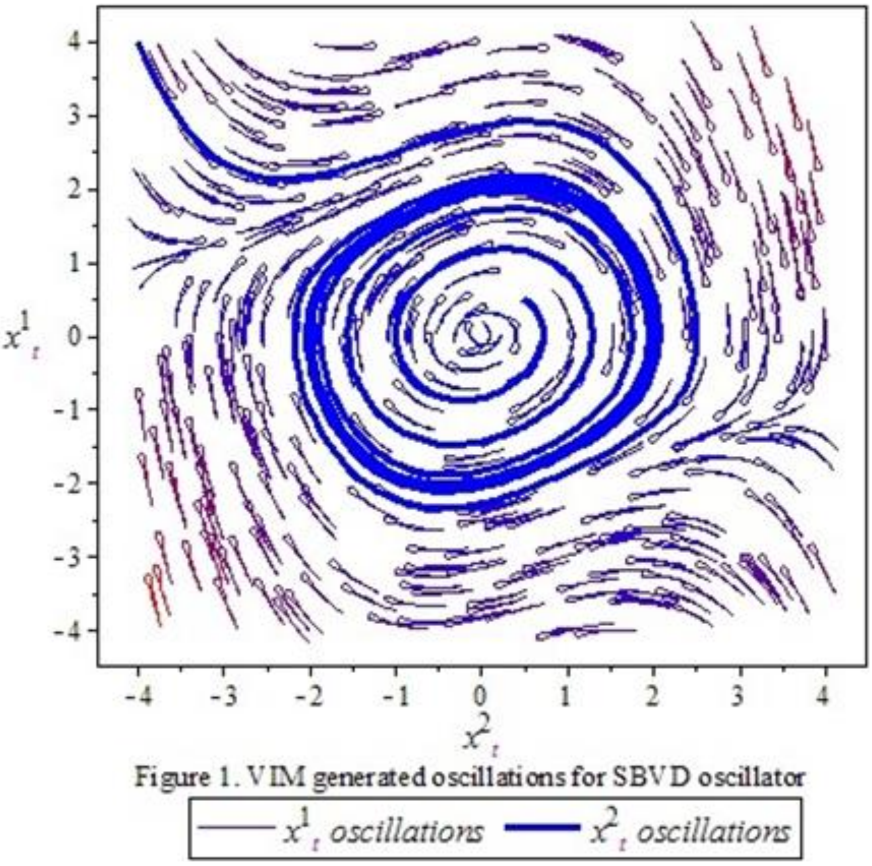

Figure 1. VIM generated oscillations for SBVD oscillator.

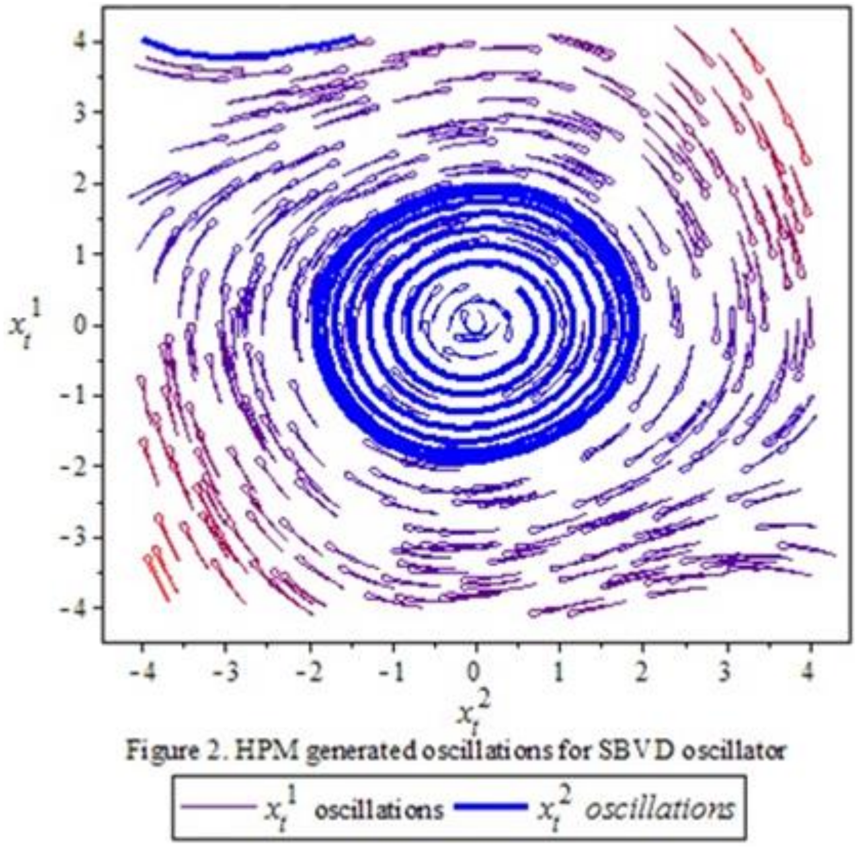

Figure 2. HPM generated oscillations for SBVD oscillator.

CONFLICT OF INTERESTS

The authors have not declared any conflict of interest.

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