OUTER APPROXIMATION TECHNIQUE AND INTERIOR POINT METHOD FOR CONVEX SETS

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Many real life problems involving management decision or policy making over limited available resources are usually formulated as optimization problems. This work focuses on two major techniques of obtaining solutions in global optimization: Outer Approximation Method (OAM) and Interior Point Method (IPM). Special consideration is made on constraint dropping strategy over a polyhedral set which is a technique of OAM and compares it with interior point method. Computational steps show that IPM performs very well due to its gradient-based property but the same optimal solution. The OAM technique discussed allows nonlinear cuts and unbounded feasible sets.

Key words: Interior point method, fixed point theorem, outer approximation, neighbourhood generation.

INTRODUCTION

This current work considers two special techniques of handling a bounded feasible set known as Polyhedral sets. There are several methods for global optimization such as the Branch and Bound methods, Neighbour generation method; the Fixed Point method, Interior Point Method(IPM), different gradient techniques, especially the approximate gradient method and of course, the OAM.. The OAM of feasible set involves a sequence of relaxed sets which could be broken down into simpler subsets or the inner approximation, which is also an aspect of the OAM. Series of works have been done in this field. Hence, this present work brings together some of these different approaches of OAM and IPM. IPM came about as need to improve the existing solution technique.

Since the pioneering works of Cheney and Goldstein (1959) and then Golmory (1957) on the algorithm for Integer Programming solutions, many researchers have proposed a number of working algorithms. Horst and Thoai (1984), Benson (1996), Horst et al. (1989a and 1989b) were on the front burner to promote OAM by Polyhedral convex set. However, Karmarkar (1984a)'s interior point generation LP problems came on board. His contributions also showed that the optimal solutions so obtained were global. Rosen (1988) focused more on global minimization for concave functions. With continuous improvement, OAM has developed into a basic tool in combinatorial optimization. Horst et al. (1989a, b) presented a large variety of methods. IPMs as proposed in Karmarkar (1984a, b) have also been highlighted in this work. It is a method that works in opposite direction with OAM but arrives at the same destination. Some of these optimization problems can be handled through the duality approach. A variant OAM developed by Benson(1996) has been considered. The cutting plane method laid the foundation and then improvement in recent algorithms which are more efficient. Kelly (1960), Topkis (1970) and Veinott (1997) have all contributed in different dimensions as early arrivals. Recent contributions are now channeled towards bilevel Mixed integer programming problems (Kleinert and Schmidt, 2019; Lozano and Smith, 2017). Fletcher and Leyffer(1994) administered OAM in solving mixed integer nonlinear programs. Several numerical illustrations have been given in Ehrgott et al. (2007).

In the angle of IPM, Arbel (1993, 1994) and

Arbel and Oren (1996) purported that obtaining the interior multi-objective LP is better through approximate gradient method. An LP model formulated as an OAM having combinatorial lower bound problem in terms of machine problems has been studied by Goemans et al. (2002) and Dyer and Wolsey (1990) with a comparison. This comparison corresponds to a technique of assignment problem where variables show which activity is being processed. Another relaxation method was proposed by Fukushima (1983) with an algorithm that solves each stage of the constraint problems independent of the solution from the previous stage. Plaxco et al. (2014) also contributed by using polynomial time completion and it minimized the mean busy time of preemptive time indexed formulation in Goemans et al.(2002). However, Cornuejols (2008) explained how to recognize valid inequalities that would be used to generate interior of multi-objective linear program. It surveys the method of developing a framework for dual mixed integer linear program.

In many applications, the ideas portrayed are to solve the resulting optimization problem and show that the results obtained are globally optimal for the problems arising from the model formulations. Next, the workings of the two techniques are presented.

Let us consider the optimization problem below

$$minf(x) \quad \text{subject to } x \in D \tag{1}$$

D is a compact set of n-tupple decision elements

Outer approximation method

The general technique of OAM is obtained by reformulating (1) as a minimization of a sequence of simpler relaxed sub-functions contained in the compact set, D. that is,

$$\underbrace{min}_{k} f_k(x_k) \quad \text{subject to } x_i \in D_k,\ i = 1,2,\ldots,n \tag{2}$$

Such that $f: \mathbb{R}^n \to \mathbb{R} \supset D_1 \supset D_2 \supset \cdots \supset D_k$

Also, $\min f_k(D_k) \to \min f(D)$. Moreover, the sets of $D_k$belong to J_k, where J_k is a family of closed convex sets with the following characteristics:

(i) The sets, of: $D_k \subset \mathbb{R}^n$are closed for all k and any problem (2) having D_k.r has a solution can be solved by algorithms available.

(ii) For any $D_k \in J_k$contained in D and any point $x_i \in D_k$, without D, can be determined by defining the special constraint function $(i\ell_k: \mathbb{R}^n \to \mathbb{R}$such that

$$\left.\begin{array}{l} \ell_k(x) \le 0, \forall x \in D \\ \ell_k(x_k) > 0, \{x \in D_k: \ell_k(x) \le 0\} \in J \end{array}\right\} \tag{3}$$

The properties $\ell_k(x) \le 0, \forall X \in D \text{ and } \ell_k > 0$ denote that $\{x \in \mathbb{R}^n: \ell_k(x) \le 0\}$ is dense and strictly separate $x_k \in D_k/D$ from D. The process of breaking down the original constraint into sub-problems in form of additional constraints is known as constraint dropping strategy over a polyhedral function. The constraint dropping idea comes from the behavior of the additional constraint $\ell_k(x_k) > 0$ which cuts off a subset of D_k. Recall also that $D \subset D_k \forall k$, meaning that no part of D is cut-off. Each $D_k$consists of outer approximation of D.

Consequently, since $D_k \supset D_{k+1} \supset \cdots \supset D$, we have that:

$$minf(D_k) \le f(D_{k+1}) \le \ldots \le minf(D) \text{and} \quad x_k \in D \Longrightarrow x_k \in argminf(D) \tag{4}$$

Given the initial set D_1containing the feasible region D, a successive sets of $D_k, k = 1,2, \ldots$ are constructed iteratively in a way that $D_k \supset D_{k+1}$and $x_k \notin D_{k+1}$. If $x_k \in D$, then x_k is a global solution. We now present results on the principles of Outer approximation method. The result is due to Horst and Tuy (1993).

THEOREM 1

That (i) the sequence ℓ_j is a lower semi-continuous for each j=1,2,3,…

(ii) each convergent subsequence$\{x_j\} \subset \{x_i'\}$ satisfying $x_i \to x^0, i \to \infty$ contains a subsequence

$\{x_j\} \subset \{x_i\}$ such that $\lim_{j\to\infty} \ell_j(x_j) = \lim_{j\to\infty} \ell_j(\bar{x})$ and $\lim_{j\to\infty} \ell_j(\bar{x}) = 0 \Rightarrow \bar{x} \in D$. Then every limit point of the sequence $\ell_j(x_j)$ is in D and also solves (3)
Proof: see Cheney and Goldstein(1959)

SOLUTION OF THE NEW VERTICES

Let the current set of inequalities be relaxed as in (3)
$\ell_k(x_k) > 0, \{x \in D_k: \ell_k(x) \le 0\} \in J_k$

k is finite set of indices and $\ell_k(x) \le 0$ the new constraint defining the extended domain

$$D_{k+i} = \{x \in \mathbb{R}: \ell_k(x) \le 0, i \in k \cup \{k\} \quad (4)$$

If
y_k *and* y_{k+1} *are the vertex sets of* D_k *and* D_{k+1} respectively with the extreme directional as u_k *and* u_{k+1} .
Then standardizing , we reformulate additional constraints as

$$\ell_k(x) = ax + x^k, \ x^k \in \mathbb{R} \quad (5)$$

And define y_k and u_k as below

$$y_k^+ := \{y \in y_k: \ell_k(y) > 0\}, y_k^- := \{y \in y_k: \ell_k(y) < 0\} \quad (6a)$$

$$u_k^+ := \{y \in y_k: a_k u > 0\}, \quad u_k^- := \{u \in u_k: a_k u < 0\} \quad (6b)$$

Converting the inequalities in 6a and b to equations by equating both the positive and negative sides to zero, a solution of y_k *is used to obtain* y_{k+1}. Similarly, a solution of u_k *is used for* u_{k+1} The set of equations are solved using LP technique. The vertices obtained are points in the Polytope D_k and intersect with unbounded edge of D_k. The extreme directions are also determined and the redundant constraints are eliminated in such a way that their removal does not affect or change the solutions obtained. There are several variants of OAM. Consider the following solution steps.

METHODOLOGY

This method works perfectly for linear programming problem(LPP). It is also suitable for general continuous optimization and applicable to nonlinear programs. Like the bundle method that is popularly used for non differentiable convex minimization, the results are the same as OAM.
Let us consider Equation 2 with slight modification as:

$min f_d(x)$, subject to $x \in D$..(I)
and $min \sum_{j=1}^{n} f(y^j)\gamma_j$,
subject to $\sum_{j=1}^{n} (Ay_j)\gamma_j \le b$...(II)
$D_k \sum_{j=1}^{n} \gamma_j = 1$.

} BB

Let the feasible region of (II) of BB be denoted as D’, A is an mxn matrix and $b \in \mathbb{R}^m$. Let$y^0, y^1, .., y^n$ be vertices of n-simplex, $Y \subset E$. Let also $f_d: \mathbb{R}^n \to \mathbb{R}$ be a convex envelope of f *on* Y, then the following conditions hold;

(i) $\dot{\gamma} \in D'$ *while* $\dot{x} = \sum_{j=0}^{n} \dot{\gamma}_j y^j \in D$ *and* $f_d(\dot{x}) = \sum_{j=0}^{n} (y^j)\dot{\gamma}_j$.
(ii) if $\dot{\gamma} \in D'$ is an extreme point of D' then $\dot{x} = \sum_{j=0}^{n} \dot{\gamma}_j y^j$ is an extreme point of D.
(iii) that the optimal solutions of the two models in BB are equal or unique. That is, if $x^*$is an optimal solution to (I) of BB then $\gamma^* \in D'$ is an optimal solution to II of BB.
We can show that the above conditions are true by
(1) Assume that $\dot{\gamma} \in D$ and $\dot{x} = \sum_{j=0}^{n} \dot{\gamma}_j y^j \in D$. Since

$$\sum_{j=1}^{n} (Ay_j)\gamma_j \le b, \text{ then } A\dot{x} \le b \text{ such that } \dot{x} \in D. \quad (7)$$

This implies that f_d in BB is an affine function and also satisfies $f_d(y^j) = f(y^j)$, j=1,2,…,n. Hence

$$,f_d(\dot{x}) = f_d(\sum_{j=0}^{n} \dot{\gamma}_j y^j) = \sum_{j=0}^{n} \dot{\gamma}_j f_d(y^j) = \sum_{j=0}^{n} \dot{\gamma}_j f(y^j). \quad (8)$$

(2) Suppose $\dot{x}$ is not an extreme point of D,

given $x^1, x^{\dot{2}} \in D$ distinct from $\dot{x}$ and some scalars

$$\beta \in (0,1), \dot{x} = \beta x^1 + (1-\beta)x^2 \quad (9)$$

In the same way for $\gamma^1, \gamma^{\dot{2}} \in D'$ we have that $x^i = \sum_{j=0}^{n} \gamma^i y^j,\ \ i = 1, 2$ then

$$\dot{x} = \beta \sum_{j=0}^{n} \gamma_j^1 y^j + (1-\beta) \sum_{j=0}^{n} \gamma_j^2 y^j = \sum_{j=0}^{n} (\beta\gamma_j^1 + (1-\beta)\gamma_j^2) y^j \epsilon D' \quad (10)$$

By the uniqueness of γ^i, there exists no $\bar{\gamma} \in D'$ also no $\bar{\gamma} \neq \dot{\gamma}$ exists.
Hence,
$\dot{\gamma} = \beta\gamma^1 + (1-\beta)\gamma^2 \therefore \dot{\gamma} = \gamma^1 = \gamma^2$
implying that $\dot{\gamma}$ is an extreme point of D'.
(3)To show that condition (iii) above hold, we assume that x^* is an optimal solution to problem (I) of BB.$\gamma^* \in D'$is a unique element of D' satisfying $x^* = \sum_{j=0}^{n} \gamma_j^* y^j$. Moreover, if $\bar{x} = \sum_{j=0}^{n} \bar{\gamma} y^j \in D$ then x^* is an optimal solution to problem I of BB. Thus, $f_d(x^*) \leq f_d(\bar{x})$
This is possible by initiating the following steps

a) Choose an initial n-simplex v_i, containing D ($D \subset V$)
b) Create a partition of the initial n-simplex to obtain n sub- division of n-simplex. Each partition comprises n-sub-simplex with lower bound(LB).
c) Minimize LB over the interval generated from the previous iteration. This is used in the current iteration.

INTERIOR POINT METHODS

This technique is especially useful in solving primal-dual linear programming problem especially of the multi-objective type. With this approach, a single objective problem is reformulated into multi-objective form by deriving the gradient of the utility function. Projecting the approximate gradient generates the interior steps. Following the interior step from current iteration to a new projection, a number of variants have been introduced by Arbel (1993, 1994). The Karush-Kuhn Tucker (KKT), the dual problem and the log barrier function are some of the options of IPM. Let the primal programming problem be given as:

$$\begin{aligned} &min\ C_j^T x \\ &subject\ to\ Ax = b, x \geq o, \end{aligned} \quad (11)$$

Standardizing and then obtaining the dual yields

$$\begin{aligned} &max\ b_j^T y \\ &subject\ to\ A^T y + y^i = C^i,\ y, y^i \geq o,\ \ 1 \leq i \leq m \end{aligned} \quad (12)$$

Instead of using the simplex iterative method, the derivative is obtained following the step direction.

$$dx^j = y_j^{-1} w^j(\bar{u}_j) - y_j^{-1} D dy^j, 1 \leq j \leq n, \quad (13)$$

where

$$dy^j = -(Ay_i^{-1} D A^T)^{-1} A y_i^{-1} w^i(\bar{u}_i) \quad (14)$$

$$dy^i = -A^T dy^j \quad (15)$$

$$w^i \bar{u}_i = \bar{u}_i I \ldots . D y_i I,\ \ w^i(\bar{u}) \in \mathbb{R}^n \quad (16)$$

$$\bar{u}_i = \sigma I^T D y_i I / n \quad (17)$$

And finally,

$$dx = \sum_{j=0}^{n} \lambda_j dx_j, \quad \text{and} \quad \sum_{j=0}^{n} \lambda_j = 1,\ \ \lambda > 0. \quad (18)$$

y^idenotes the additional variables resulting from either slack, surplus or artificial variables for each of the constraint equation. The dy^i acts like the directional derivative for the i constraint. From Equations 10 to 18, D and y_i are the diagonal nxn matrix representing the components of current iterates of x and y^i. Once the directional derivative is obtained, the interior step direction follows. This is synonymous to the approximate gradient. Consider the model in (11),
if $F(y) = \nabla f(y)\colon \dot{y} \in D$, f is differentiable.

$$\nabla f(\dot{y})^T (y - \dot{y}) \geq 0\ \forall y \in D \quad (19)$$

Equation 19 is called a variational inequality and monotone if

$$F: R^n \to R^n, \qquad (F(y) - F(x))^T (y - x) \geq 0, \quad \forall x, y \qquad (20)$$

Another approach is to obtain the logarithmic barrier function associated with Equation 11 as:

$$B(x, \mu) = f(x) - \mu \sum_{j=1}^{n} log\, C_j^T x\,, \qquad (21)$$

μ is a positive scalar known as the barrier parameter. The advantage is that it guarantees convergence of solution as $\mu \to 0$, (21) tends to a solution of Equation 11.

ILLUSTRATIONS

Solve the multi-objective problem with three objectives. Below are the cost matrices of the 3 objectives:

$$\begin{pmatrix} 3 & 6 & 4 & 5 \\ 2 & 3 & 5 & 4 \\ 3 & 5 & 4 & 2 \\ 4 & 5 & 3 & 6 \end{pmatrix}, \begin{pmatrix} 2 & 3 & 5 & 4 \\ 5 & 3 & 4 & 3 \\ 5 & 2 & 6 & 4 \\ 4 & 5 & 2 & 5 \end{pmatrix}, \begin{pmatrix} 4 & 2 & 4 & 2 \\ 4 & 2 & 4 & 6 \\ 4 & 2 & 6 & 3 \\ 2 & 4 & 5 & 3 \end{pmatrix}$$

Solution Example 1

For OAM the non-dominated vertices from the cost matrices are (11,11,14), (19,14,10), (15,9,17) and (13,16,11). These correspond to the facets of the domain D by

$$3y_1 + 3y_2 + y_3 = 14$$
$$-9y_1 - y_2 + y_3 = 10$$
$$2y_1 + 8y_2 + y_3 = 17$$
$$-2y_1 - 5y_2 + y_3 = 11$$

for the 4 facets respectively. D has 9 vertices (Table 1).

Example 2

$$\underbrace{min}_{k=1,2} f_k = y_1^2 + y_2^2 + 2$$
$$subject\ to\ \ (y_1 - 2)^2 - y_2 \leq 0$$
$$y_1 \geq 0$$
$$y_1 - y_2 - 3 \leq 0$$
$$y_1 - 1 \geq 0$$
$$y_2 - 1 \geq 0$$
$$y_1 - y_2 \geq 0$$
$$0 \leq y_1 \leq 4, \qquad 0 \leq y_2 \leq 4$$

Table 1. The vertices and the corresponding facets with values obtained.

Vertices of D			Facets
y_1	y_2	y_3	
1	**0**	**11**	$y_1 = 11$
0	**1**	**9**	$y_2 = 9$
0	**0**	**10**	$y_3 = 10$
$\frac{1}{3}$	$\frac{2}{3}$	**11**	$\frac{1}{3}y_1 + \frac{2}{3}y_2 = 11$
$\frac{3}{5}$	**0**	**12½**	$\frac{3}{5}y_1 + \frac{2}{5}y_3 = 12\frac{1}{5}$
0	$\frac{4}{7}$	$12\frac{2}{7}$	$\frac{4}{7}y_2 + \frac{3}{7}y_3 = 12\frac{2}{7}$
$\frac{1}{7}$	**0**	$11\frac{2}{7}$	$\frac{1}{7}y_1 + \frac{6}{7}y_3 = 11\frac{2}{7}$
0	$\frac{3}{5}$	**12**	$\frac{1}{5}\frac{3}{5}y_2 + \frac{2}{5}y_3 = 12\frac{1}{5}$
$\frac{11}{61}$	$\frac{16}{61}$	$12\frac{41}{61}$	$\frac{11}{61}y_1 + \frac{16}{61}y_2 + \frac{34}{61}y_3 = 12\frac{41}{61}$

Method 2 of OAM to obtain the solution of $(y_1, y_2) = [1,1]$. The optimal solution has upper bound as 4 .

Example 3

$$minimize\ f(x) = \sum_{i=1}^{\infty} logx_i \geq 0\ subject\ to\ Ax = b$$

Solution: feasible point IPM: $g = \nabla \mathrm{f(x)}$, $\mathrm{H} = \nabla^2 f(x)$

$$and\ g = \begin{bmatrix} \frac{-1}{x_1} \\ \frac{-1}{x_n} \end{bmatrix}, \qquad H = diagonal \left[\frac{1}{x_1^2} \quad , \cdots, \quad \frac{1}{x_n^2} \right]$$

The Hessian is positive definite,

$$\begin{bmatrix} H & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} d \\ u \end{bmatrix} = \begin{bmatrix} -g \\ 0 \end{bmatrix},\ the\ first\ row\ is\ Hd + A^T u = -g$$

$$\Rightarrow g - H^{-1}(g + A^T u).\ The\ second\ row Ad = 0, \Rightarrow AH^{-1}(g + A^T u) = 0$$

$$\Rightarrow AH^{-1}A^T\ is\ invertible.\ Hence, u = -AH^{-1}A^T)^{-1} AH^{-1} g,$$

$$H^{-1} = diagonal\{x_1^2, \ldots x_n^2\}$$

The matrix $AH^{-1}A^T$ is known as Schur complement of H.

Explanation

In Example 1, the dual simplex was used to solve the reformulated linear cuts in OAM while the constraints system was decomposed from the primal into its dual following the steps of IPM to obtain solution. The results obtained showed the same optimal solution. The number of constraints in the optimal table was 16 in 16 unknown, with 4 vertices, 3 facets. It was observed that the dual is faster if the decomposed constraints have fewer facets. Examples 2 and 3 were solved using OAM and IPM respectively.

Conclusion

The OAM came into existence to overcome the problems of extreme point ranking through the additional constraints (affine cuts) introduced which also guarantee convergence. The cutting plane method was the first algorithm developed for Integer programming and later modified to obtain the Branch and Bound method. Thus, it gave an insight for more efficient algorithms. OAM is an improvement of these techniques. The advantage attached to OAM is that the steps involved are finite; the solution is exact and globally optimal. The IPM also guarantees convergence and is affine invariance. For the OAM, as the number of linear constraints increases at each stage of the iteration, computational effort increases. By proceeding with the solution, redundant constraints are dropped while a finite number of the relevant cuts are retained. Apart from this, at each stage again, the localized sets are replaced with bounded set as an additional constraint. If the function f in Equations 11 or 19 is nonlinear, then the localized sets are no longer Polyhedral. The common factor is that the two techniques can be utilized to solve both linear and nonlinear programming. On the other hand, the IPM is not too efficient especially for large convex problem.

CONFLICT OF INTERESTS

The author has not declared any conflict of interests.

REFERENCES

Arbel, A. (1993). An Interior Multi-objective Linear Programming Algorithm. *Computer and Operations Research*, 20(7), 723-735.

Arbel, A. (1994). Anchoring Points and Cones of Opportunities in Interior Multi-objective linear Programming Problems. *Journal of Operations Research Society*, 45(1),83-96.

Arbel, A and Oren, S.S (1996). Using Approximate gradients in developing interior Primal-Dual Multi-objective linear Programming Algorithm. *European Journal of Operations Research*, 89, 202-211.

Benson, H.P. (1996). Deterministic Algorithms for Constrained Concave Minimization: A Unified Critical Survey, Naval Research Logistics, 43, 765 – 795

Cheney, E.W. and Goldstein .A.A (1959). Newton's Method for Convex Programming and Tchebycheff Approximation Numerische Mathematik, l, 253 – 268.

Cornuejols, G (2008). Valid Inequalities for Mixed Integer linear Programs. *Mathematical Programming Series* B, 112, 3-44.

Dyer, M and Wolsey, A. (1990). Formulating the Single Machine Sequencing Problem with release dates as Mixed Integer Program. *Discrete Applied Mathematics*, 26, 255-270.

Ehrgott, M.; Lohne, A and Shao, L. (2007). *A Dual Variant of Benson's Outer Approximation Algorithm*. University of Auckland Press, New Zealand.

Fletcher, R and Leyffer, S (1994). Solving Mixed Integer Nonlinear Program by OAM. *Mathematical Programming*, 66(1): 327-349.

Fukushima, M. (1983). An Outer Approximation for Solving General Convex Program. *Operations Research*, 31(1): 101-113.

Goemans, M.X; Queyranne, M; Schulz; A.S.; Skutella, M and Wang, Y. (2002). Single Machine Scheduling with release dates. *SIAM Journal on Discrete Mathematics*, 15(2):165-192.

Golmory, R.E. (1957). Outline of an algorithm for integer solutions to linear programs. *.Bulletin of American Mathematics Society*, 64: 275 - 278

Horst,.R., and Tuy, H. (1993). *Global optimization: Deterministic approaches.* Springer, Berlin.

Horst.R., and Thoai. N.V. (1994). Constraint decomposition algorithms for Global Optimization. *Journal of Global Optimization*, 5: 333 – 348

Horst. R., Thoai, N.V. and Tuy, H. (1989a). Outer approximation by polyhedral convex sets. *Operations Research Spektrum,* 9: 153 - 159

Horst. R., Thoai, N.V. and Tuy, H. (1989b). On an Outer Approximation Concept in Global Optimization. *Optimization* , 20: 255 - 264

Karmarkar, N (1984a). A new Polynomial-time Algorithm for the Linear Programming. *Combinatorica* , 4: 373-395.

Karmarkar, N (1984b). A new Polynomial-time Algorithm for the Linear Programming. Proceeding of 16th Annual ACM Symposium on Theory of Computing, Washington, D.C.

Kelley. J.E. (1960). The Cutting-Plane Method for Solving Convex Programs. *SIAM Journals,* 8: 703 - 712

Kleinert, T and Schmidt, M. (2019). Global Optimization of Multi-level Electricity Market Models including Network design and graph Partitioning. *Discrete Optimization*, 33:43-69.

Lozano, L and Smith, J . C. (). (2017). A value-function-based Exact Approach for the bi-level Mixed-integer Programming Problem. *Operations Research*, 65(3):768-786.

Plaxco, J., Valdes, A and Stojko, W. (2014). Interior Point for LP. CHE, 345,Spring, 2014.

Rosen, J.B (1988). Global Minimization of a Linearly constrained concave Function by Partition of Feasible Domain. *Mathematics of Operations Research*, 8: 215-230.

Topkis. D.M. (1970). Cutting-Plane Methods without Nested Constraint Sets". *Operations Research*, 18: 404-413

Veinott. A.F (1967). The Supporting Hyperplane Method for Unimodal Programming. *Operation Research,* 15: 147 – 152.