APPROXIMATE SOLUTION OF NEWELL-WHITEHEAD-SEGEL EQUATION USING ITERATIVE METHODS WITH MAMADU-NJOSEH POLYNOMIALS

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This study considered the variation iteration method (VIM) and the homotopy perturbation method (HPM) for solving the Newell-Whitehead-Segel equation (NWSE). For this purpose, the Mamadu-Njoseh orthogonal polynomials were used as basis functions. Two cases of the NWSE were considered by variation of some parameters and with their analytic solutions given and formulated for numerical computations via VIM and HPM. Interesting results were obtained in the course of implementation and observed that the rate of convergence of solutions are controlled by the same parameters for both VIM and HPM iterative schemes. Further investigation revealed that the higher the values of these parameters, the faster the rate of convergence of the solutions for both methods. We presented the resulting numerical evidence in tables and graphs and our results compared with the analytic solutions as available in literature. Thus, we observed that the VIM and HPM have same rate of convergence for the various cases of the Newell-Whitehead-Segel equation considered. All computational frame works of this research were implemented with the aid of MAPLE 18.

Keywords: Orthogonal polynomials, Mamadu-Njoseh polynomials, Variational iteration method, Homotopy perturbation method

1. Introduction

In order to describe the behavior and effect of some phenomena in the different fields of science as well as engineering, functional equations such as linear and nonlinear partial differential equations, integral and integro-differential equations and stochastic equations are used. The non-linear evolution equations (NLEE) play an important and significant role in modeling various physical phenomena related to solid state physics, fluid mechanics, plasma physics, population dynamics, chemical kinetics, nonlinear optics, protein chemistry and many other fields of applied science. Several systems are modeled by partial differential equations and most of them are nonlinear. Investigation of the exact solutions of nonlinear partial differential equations plays a key role in the study of nonlinear physical phenomena. Most of the equations do not have analytical solutions but can be solved by semi-analytical methods. To obtain exact solution of nonlinear differential equations, semi-analytical methods such as the Variational Iteration Method (VIM) and Homotopy Perturbation Method (HPM) with orthogonal polynomials as basis functions can be considered. They are powerful algorithms in solving various kinds of linear and nonlinear equations. The numerical and analytical approximations of Partial Differential Equations (PDEs) have always been an active field of study in Physics, Mathematics and Engineering. Many researchers have proposed several approaches to solve different PDEs. For instance, He (1999) solved some wave equations with the Homotopy Perturbation Method (HPM). Akinlabi and Edeki (2017) also solved initial value wave-like models using the modified

Differential Transform Method (DTM). Likewise, Edeki *et al.* (2016) considered the numerical and the analytic solutions of time-fractional linear Schrodinger equations. Solving nonlinear systems is an important task in mathematical analysis and applications. One of the important amplitude equations is the Newell-Whitehead-Segel Equation (NWSE), written as

$$u_T = ku_{xx} + au - bu^q,$$

Where a, b and k are real numbers with k > 0, and q is a positive integer, which describes the appearance of the stripe pattern in two dimensional systems. Moreover, this equation was applied to a number of problems in a variety system, e.g., Rayleigh-Bernard convection, Faraday instability, nonlinear optics, chemical reactions and biological systems (Akinlabi and Edeki, 2017). The Newell-Whitehead-Segel equation is a NLEE for Bernard's problem (Manaa, 2011). Bernard's problem is a hydrodynamic problem in which water contained between two plates is heated from below. It exhibits patterns like rolls, hexagons or rectangles if the bifurcation parameter, which is related to temperature differences between the plates, is above a certain threshold. The Newel-Whitehead-Segel Equation (NWSE) is one of the significant concepts of pattern formation theory and sometimes it is called only Newell-Whitehead Equation or amplitude equation. These equations have wide applicability in mechanical and chemical engineering, ecology, biology and bioengineering. Many stripe patterns, e.g., ripples in the sand, stripes of seashells, appear in a variety of spatially extended systems which can be specified by a set of equation called amplitude equations. In recent years, different methods have been utilized to solve NWS equation. Pue-on (2013) applied Laplace Adomian Decomposition Method for Solving Newell-Whitehead-Segel Equation. Saravanan and Magesh (2013) used differential transform method, and Arqub *et al.*, (2014) used a new analytic iterative technique to obtain a generalized wave solution to NWS equation. Ezzati and Shakibi (2011) solved the Newell-Whitehead equation using the Adomian decomposition and multi-quadricquasi-interpolation methods. They concluded that the Adomian decomposition and multi-quadricquasi-interpolation methods are reasonable methods to solve the Newell-Whitehead equation with acceptable accuracy. However, in this study, we will restrict our analysis to the use of orthogonal polynomials as basis function for deriving approximate solutions to Newell-Whitehead-Segel Equations using the variational iteration method and the homotopy perturbation method. There exist several orthogonal polynomials such as Chebyshev, Hermite, Laguerre, Jacobi, Mamadu-Njoseh polynomials etc. The Mamadu-Njoseh polynomial, developed in 2016 (Mamadu and Njoseh, 2016a) will be adopted as basis functions since it tends to converge faster than most of the polynomials in existence. These polynomials have been proven to be reliable, accurate and effective in both the analytic and numerical process.

2. Methods of Solution

2.1 The Orthogonal polynomials

The orthogonal polynomials are class of polynomials $p_n(x)$ defined over a range $[a, b]$ that obey the orthogonality relation (Mamadu and Njoseh, 2016b)

$$\int_a^b w(x)\varphi_m(x)\varphi_n(x)dx = h_r\partial_{mn} \quad (1)$$

With the Kronecker delta δ_{mn} defined as

$$\delta_{mn} = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases},$$

where the weight function $w(x)$ is continuous and positive on $[a, b]$ such that the moment

$$\mu = \int_a^b w(x)x^n(x)dx, \qquad n = 0,1.2 \dots. \tag{2}$$

exist.
Then the integral

$$\langle \varphi_m, \varphi_n \rangle = \int_a^b w(x)\varphi_m(x)\varphi_n(x)dx, \tag{3}$$

Denotes the inner product of polynomial φ_m and φ_n. The interval $[a, b]$ is called the interval of orthogonality. This interval needs not to be definite. If $h_n = 1$ for each $n \in (0,1,2 \dots)$ the sequence of the polynomial is called orthonormal, and if $\varphi_n = k_n x_n$ plus lower order terms with $k_n = 1$ for each $n \in (0,1,2 \dots)$ the polynomials are called monic.

For orthogonality,

$$\langle \varphi_m, \varphi_n \rangle = \int_a^b w(x)\varphi_m(x)\varphi_n(x)dx = 0, \; m \neq n, \; x \in [-1,1] \tag{4}$$

2.2 The Mamadu-Njoseh Polynomials (Njoseh and Mamadu, 2016)

The realization of the Mamadu-Njoseh polynomials with $w(x) = 1 + x^2, \; -1 \leq x \leq 1$, were based on these three properties:

(i) $\varphi_n(x) = \sum_{r=0}^{n} C_r^{(n)} x^r$

(ii) $\langle \varphi_m(x), \varphi_n(x) \rangle = 0, m \neq n$

(iii) $\varphi_n(x) = 1$

Where $\varphi_n(x), i = 0,1,2,3 \dots.$ are orthogonal polynomials. Thus, the first six Mamadu-Njoseh polynomials are given below:

$$\varphi_0(x) = 1$$
$$\varphi_1(x) = x$$
$$\varphi_2(x) = \frac{1}{3}(5x^2 - 2)$$
$$\varphi_3(x) = \frac{1}{5}(14x^3 - 9x)$$
$$\varphi_4(x) = \frac{1}{648}(333 - 2898x^2 + 3213x^4)$$
$$\varphi_5(x) = \frac{1}{136}(325x - 1410x^3 + 1221x^5)$$
$$\varphi_6(x) = \frac{1}{1064}(-460 + 8685x^2 - 24750x^4 + 17589x^6)$$

2.3. Variational Iteration Method

This method treats both linear and non-linear equations the same without any unrealistic assumption that converges faster to the exact solution by successive approximation of the analytic solutions. The basic concept of the VIM is to construct a correctional for the problem involved (Mamadu and Njoseh, 2016c)

Let the generalized form of a differential equation be given as

$$L[u(x)] = g(x), \; u(a_1) = a, \; u(a_2) = b, \tag{5}$$

where L is a differential operator, $u(a_1) = a, \; u(a_2) = b,$ are boundary or initial conditions, $u(x)$ is the unknown function and $g\,(x)$ is the source term.

$$u_{i+1}(x) = u_i(x) + \int_0^x \lambda(s)\left(L(u_i(s)) - g(s)\right)ds, \quad i \geq 0, \tag{6}$$

where λ is a general Lagrange's multiplier. This is estimated using the formula (Njoseh and Mamadu, 2017a and b) as

$$\lambda(s) = (-1)^n \frac{(s-x)^{(n-1)}}{(n-1)!}, \tag{7}$$

where n in (7) is the order of the derivative.

2.3.1 Convergence Analysis of the Variational Iteration Method

Generally, one iteration leads to high accurate solution by variational iteration method if the initial solution is carefully chosen with some unknown parameters. If we begin with $u_0(x,t) = u(x,0)$, a series solution is obtained. The sufficient conditions are presented to guarantee the convergence of VIM, when applied to solve NWSEs, where the main point is that we prove the convergence of the recurrence sequence, which is generated by using VIM.

Theorem 3.1. Sufficient Condition (Njoseh and Mamadu, 2016)

Define u_t by $D(u)$ in equation (2.1) such that $D(u) = ku_{xx} + au - bu^q$. Then, VIM convergences if the following conditions are satisfied:

(i) $D(u) - D(v), u - v) > a||u - v||^2, a > 0, u, v \in H$, where H is a Hilbert space.
(ii) For$\lambda > 0$, there exist $\rho(\lambda) > 0$ such that $||u|| \leq \lambda, ||v|| \leq \lambda, u, v \in H$, then

$$(D(u) - D(v), u - v) > \rho(\lambda)\left||u - v|\right|\left|r|\right|, \qquad r \in H$$

Proof.

For $a > 0,\ \lambda, u, v \in H$, we have

$$(D(u) - D(v), u - v) = \left(ku_{xx} + au - bu^q - \mathrm{k}v_{xx} + av - bv^q\ , u - v\right),$$

where

$$D(v) = kv_{xx} + av - bv^q.$$

Applying the Schwartz inequality, we get

$$\begin{aligned}&(ku_{xx} + au - bu^q - kv_{xx} + av - bv^q, u - v)\\ &\leq a_1\|ku_{xx} + au - bu^q - kv_{xx} + av - bv^q\|\|u - v\|.\end{aligned}$$

Using the conventional mean value theorem, we obtain

$$\left(ku_{xx} + au - bu^q - kv_{xx} + av - bv^q, u - v\right) \geq \frac{1}{5}a_1\rho^2\|u - v\|^2.$$

$$(D(u) - D(v), u - v) \geq \frac{1}{5}a_1\rho^2\|u - v\|$$

Similarly, for $\lambda > 0$, there exist $\rho(\lambda) > 0$ such that that $\|u\|\lambda\|v\| \leq \lambda,\ u, v \in H$, then

$$\begin{aligned}(D(u) - D(v), r) &= (ku_{xx} + au - bu^q - kv_{xx} + av - bv^q)\\ &\leq \lambda^2\|u - v\|\|r\|\\ &= \rho(\lambda)\|u - v\|\|r\|.\end{aligned}$$

This completes the second condition.

2.4. The Homotopy Perturbation Method (HPM)

A combination of the perturbation method and the homotopy method is called the homotopy perturbation method (HPM), which has eliminated the limitations of traditional perturbation methods. On the other hand, this technique has the full advantage of traditional perturbation techniques. The solution of this method is considered as the summation of an infinite series, converging very fast to the analytic solution. Given the general non-liner differential equation (He, 1999)

$$L(u) + N(u) = f(r),\ r \in \Omega \tag{8}$$

with

$$B\left(u, \frac{\partial u}{\partial r}\right) = 0, \qquad r \in \Omega,$$

where L is a linear operator, N, a nonlinear operator, r is the reaction parameter and $f(r)$ is a known analytic function, B is a boundary operator, r is the boundary of the domain Ω. If we set

$$L(u) + N(u) = A(u), \tag{9}$$

then, we will have

$$A(u) - f(r) = 0,\ r \in \Omega, \tag{10}$$

3. Numerical Perspective

Recall that the Newell-Whitehead-Segel equation can be rewritten as

$$u_t = ku_{xx} + au - bu^q \tag{11}$$

$$u(x,0) = u_0 \tag{12}$$

where a, b and k are real parameters and $k > 0$ and $q > 0$.

3.1 VIM for NWSE

Let an approximate solution of the form

$$u_0(x,t) = \sum_{i=0}^{N} a_i \varphi_i(x) \tag{13}$$

Be given where $\varphi_i(x)$ are the Mamadu-Njoseh polynomials, $a_i, i = 0,1,2,\ldots$, are constants to be estimated and N is the degree of approximation. Now, the first step in the execution of the VIM for NWSE is the construction of correctional functional. Hence a correction functional for (12) is given as

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(x,s)\left(\frac{\partial u_n(x,s)}{\partial s} - k\frac{\partial^2 u_n(x,s)}{\partial x^2} - \mathrm{a}u_n(x,s) + \mathrm{b}u_n(x,s)\mathrm{q}\right) ds,\ n \geq 0. \tag{14}$$

3.1.1 Determination of the Initial Approximation

To kick-off the iterative scheme (14), we require an initial. To this end, the initial approximation can be obtained by equating (12) and (13) at $t = 0$, that is,

$$\sum_{i=0}^{N} a_i \varphi_i(x) = u_0(x,0). \tag{15}$$

Now, let $N = 3$ in (15), and collocating orthogonally at $\varphi_4(x)$; that is,
(.3676425560, −.3676425560, .8756710201, −.8756710201);
via MAPLE 18 to obtain the system

$$Ax = b, \tag{16}$$

where

$$A = \begin{bmatrix} 1 & 0.6838212780 & 0.1126859003 & -0.3355426813 \\ 1 & 0.3161787220 & -0.500516929 & -0.4806191157 \\ 1 & 0.9378555100 & 0.799227393 & 0.6215028820 \\ 1 & 0.0621644900 & -0.6602259603 & -0.1112234381 \end{bmatrix},$$

$$\underline{x} = (a_0, a_1, a_2, a_3)^T,$$

$$\underline{b} = (u_0, u_1, u_2, u_3)^T.$$

Solving (16) via Gaussian elimination method for the unknowns and substituting back into (13) to obtain the initial approximation.

3.2. HPM for NWSE

In view of the HPM discussed in section, let start off by constructing the homotopy p for (11) as follows:

$$\frac{\partial u}{\partial t} = p\left(k\frac{\partial^2 u}{\partial x^2} + au - bu^q\right) \tag{17}$$

Let assumed that the approximation to (11) be given as

$$u_0(x,t) = p\left(\sum_{i=0}^{N} a_i \varphi_i(x)\right) \tag{18}$$

Substituting (18) into (17), and comparing like powers of p on both sides, we have,

$$p^0: \frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0$$

$$p^1: \frac{\partial v_1}{\partial t} - \frac{\partial u_0}{\partial t} = k\frac{\partial^2 v_0}{\partial x^2} + \mathrm{a}v_0 - \mathrm{b}{v_0}^q$$

$$p^2: \frac{\partial v_2}{\partial t} = k\frac{\partial^2 v_1}{\partial x^2} + v_1(\mathrm{a} - \mathrm{b}v_0) - \mathrm{b}v_0 v_1$$

$$p^3: \frac{\partial v_3}{\partial t} = k\frac{\partial^2 v_1}{\partial x^2} + v_2(\mathrm{a} - \mathrm{b}v_1) - \mathrm{b}v_0 v_2 - \mathrm{b}{v_1}^q$$

$$\vdots$$

Thus, the solution can be written as

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \cdots \tag{19}$$

3.3 Numerical Simulation and Results.

Two cases of the NWSE shall be considered.

Case 1. In (11) for $a = 2, b = 3, k = 1$ and $q = 2,$ the exact solution is given as

$$u(x,t) = \frac{-\frac{2}{3}e^{2t}}{\frac{1}{3} - e^{2t}}.$$

Results for Case 1 are presented below for VIM and HPM as shown in Table 1 and Figures 1a and 1b.

Table 1: Numerical Results for Case 1 using the Second Iterate

$x, t = 0$	Exact solutio0n	VIM Solution	HPM solution	Error (VIM)	Error (HPM)
0.01	0.9990020	0.9990020	0.9990020	6.0000 E-10	6.0000 E-10
0.02	0.9980080	0.9980080	0.9980080	5.2000 E-09	5.2000 E-09
0.03	0.9970179	0.9970179	0.9970179	1.7800 E-08	1.7800 E-08
0.04	0.9960318	0.9960318	0.9960318	4.18000 E-08	4.18000 E-08
0.05	0.9950495	0.9950496	0.9950496	8.1000 E-08	8.1000 E-08
0.06	0.9940712	0.9940714	0.9940714	1.3950 E-07	1.3950 E-07
0.07	0.9930968	0.9930960	0.9930960	2.2100 E-07	2.2100 E-07
0.08	0.9921262	0.9921265	0.9921265	3.2700 E-07	3.2700 E-07
0.09	0.9911594	0.9911598	0.9911598	4.6480 E-07	4.6480 E-07
1.0	0.9901964	0.9901970	0.9901970	6.3430 E-07	6.3430 E-07

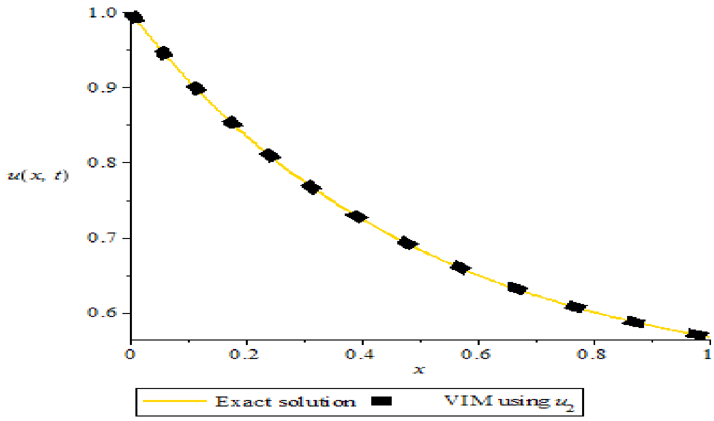

Figure 1a: VIM Versus Exact solution

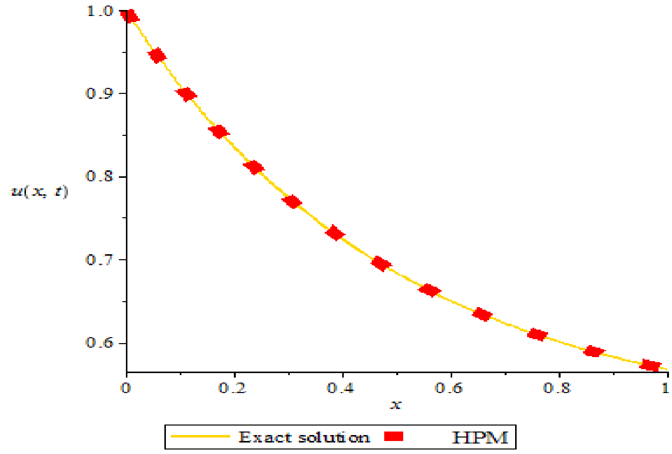

Figure 1b: HPM Versus Exact solution

Case2. In (11) for $a = 1, b = 1, \ k = 1$ and $q = 2$, the exact solution is given as

$$u(x,t) = \frac{1}{1+e^{\left(\frac{x}{\sqrt{6}}-\frac{5t}{\sqrt{6}}\right)^2}}.$$

Results for Case 2 are presented below for VIM and HPM as shown in Table 2 and Figures 2a and 2b.

Table 2: Numerical Results for Case2 using the First Iterate

$x, t = 0$	Exact solution	VIM Solution	HPM solution	Error (VIM)	Error (HPM)
0.01	0.2398993	0.2398264	0.2398264	2.7048 E-05	2.7048 E-05
0.02	0.2300151	0.2300151	0.2300151	1.9896 E-05	1.9896 E-05
0.03	0.2203548	0.2203636	0.2203636	8.8534 E-06	8.8534 E-06
0.04	0.2109249	0.2109205	0.2109205	4.3938 E-06	4.3938 E-06
0.05	0.2017316	0.2017142	0.2017142	1.7402 E-05	1.7402 E-05
0.06	0.1927799	0.1927530	0.1927530	2.6962 E-05	2.6962 E-05
0.07	0.1840743	0.1840742	0.1840742	2.9093 E-05	2.9093 E-05
0.08	0.1756184	0.1756183	0.1756183	1.9043 E-06	1.9043 E-06
0.09	0.1674150	0.1674157	0.1674157	8.7058 E-06	8.7058 E-06
1.0	0.1594662	0.1594667	0.1594667	6.0428 E-05	6.0428 E-05

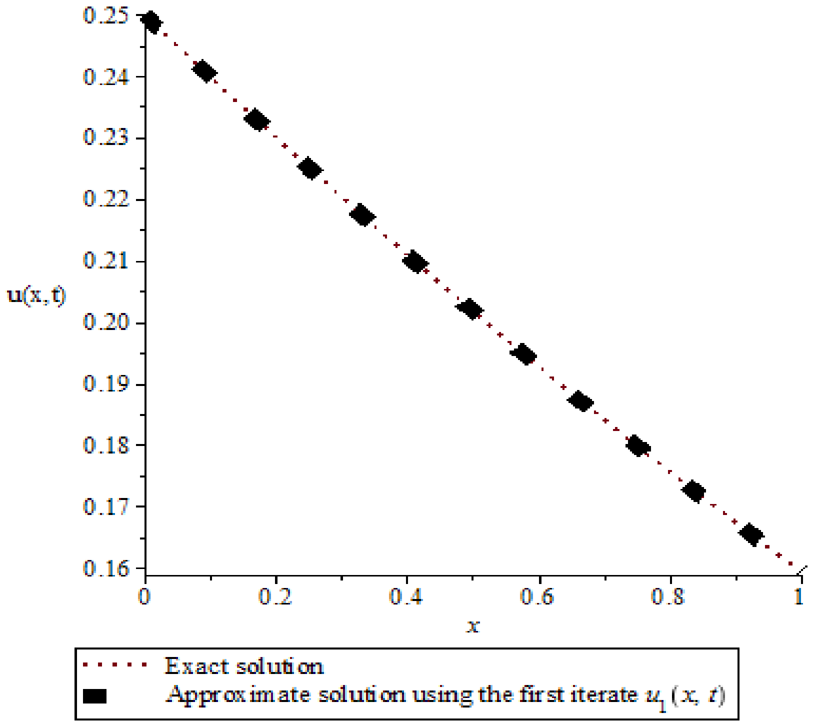

Figure 2a: VIM Versus Exact solution

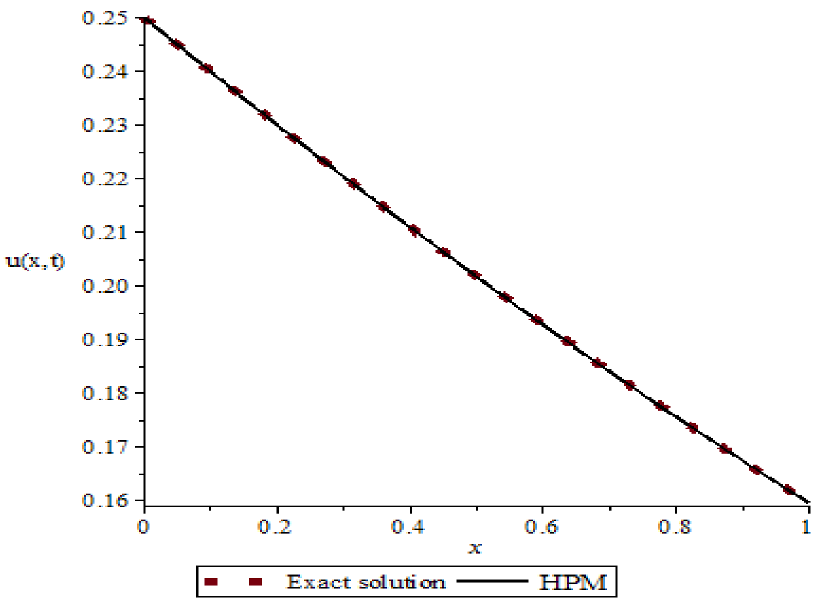

Figure 2b: HPM Versus Exact solution

4. DISCUSSION

We have obtained some fascinating results in the course of implementing the VIM and HPM via the Mamadu-Njoseh orthogonal polynomials as basis or trial functions in the approximation of Newell-Whitehead-Segel equation. We have presented resulting numerical evidence in tables and graphs with results compared with the analytics as available in literature. It was observed that the rate of convergence of solutions are controlled by the parameters a and b for all cases considered as shown in Tables 1 and 2. For emphasis, for a = 2 and b = 3, a maximum error of order 10^{-10} was obtained for case 1 which converges favorably to the exact solution for both the VIM and HPM in Table 1. Similarly, for a = 1 and b = 1, maximum error of order 10^{-6} was obtained for case 2 as shown in Table 2. This suggest that the higher the values of a and b the faster the rate of convergence of solution for both the VIM and HPM. Results are also presented graphically to highlight the rate of convergence of solution for both methods.

5. Conclusion

This study has considered the variational iteration method (VIM) and the homotopy perturbation method (HPM) for solving the Newell-Whitehead-Segel equation. So far, it has been established that orthogonal polynomials can be used as basis function in solving many mathematical models such as the Newell-Whitehead-Segel equation. It is also evident that these methods offer several advantages which includes, among others:

i. cost-effectiveness as no extra interpolation is required in other to achieve several outputs of solution;
ii. case of implementation; and
iii. efficiency and effectiveness as several outputs of solution were obtained at some grid-point.

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